THE AMERICAN MATHEMATICAL MONTHLY.

DEVOTED TO THE
SOLUTION OF PROBLEMS IN PURE AND APPLIED MATHEMATICS,
PAPERS ON MATHEMATICAL SUBJECTS, BIOGRAPHIES
OF NOTED MATHEMATICANS, ETC.

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THE

AMERICAN

MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

Vol. IV. JANUARY, 1897. No. 1.

BIOGRAPHY.

DE MORGAN.

BY GEORGE BRUCE HALSTED.

AUGUSTUS DE MORGAN is fortunately a well-known character. The world has an excellent sketch of him by his pupil Jevons in the Encyclopædia Britannica, from which I copy literally some of the preliminary biographic data here given. If I am able to add anything specifically new to this part of the paper, it is from a life-long study of his works, and many conversations about him with the great Sylvester, who knew him very intimately.

Augustus De Morgan was born in June, 1806, in India. On his mother's side he was descended from James Dodson, F. R. S., author of the Anti-Logarithmic Canon and other mathematical works of merit, and a friend of De Moivre. Our Augustus lost one eye in his early infancy, and this prevented his joining in the usual games. He read algebra "like a novel," and pricked out equations on the school pew instead of listening to the sermons.

When 16 years old he entered Trinity College, Cambridge, and studied his mathematics partly under the tuition of Airy, making many friends, including his teachers Whewell and Peacock. In 1825 he gained a Trinity Scholarship.

But De Morgan's attention was by no means confined to mathematics, and his love of wide reading somewhat interfered with his success in the mathematical tripos, in which he took fourth place in 1827, before he had completed his 21st year.

He was prevented from taking his M. A. degree, or from obtaining a fel-


A. De Morgan

lowship, by his conscientious objection to signing the theological tests then required from masters of arts and Fellows at Cambridge. Jevons says, "a strong repugnance to any sectarian restraint upon the freedom of opinion was one of De Morgan's most marked characteristics throughout life."

A career in his own university being thus closed against him, on the twenty-third of February, 1828, he was elected Professor of Mathematics in University College, London, and began lecturing at the early age of 22 years.

But the regents of this College claimed the right of dismissing a professor without assigning reasons, and acted upon this principle in dismissing the Professor of Anatomy. Immediately De Morgan resigned in protest. After the regulations were changed he was invited to resume his chair, was reappointed, and served for the next 30 years. His dislike to honorary titles led him to refuse the offer of LL. D. from the University of Edinburgh.

In 1866 a discussion arose as to the true interpretation of the principle of religious neutrality avowedly adopted by the College. De Morgan held that any consideration of a candidate's ecclesiastical position or creed or lack of creed was inconsistent with the principle. In protest against a violation of this principle he resigned a second time in a letter dated November 10th, 1866. He was always remarkably free from any touch of sordid self-interest.

As a teacher, De Morgan was particularly gifted. A voluminous writer on mathematics, he contributed essentially to those expansions of the fundamental concepts which have rendered possible the new algebras, such as Quaternions, and have generalized the whole idea of a mathematical algorithm, until now an American produces a thesis entitled "Mathematics the Science of Algorithms."

His logical work alone would give De Morgan lasting fame. Here he stands alongside his immortal contemporary, Boole. The eternally memorable year in the history of Logic was 1847, in which George Boole issued "The Mathematical Analysis of Logic, being an Essay toward a Calculus of Deductive Reasoning," von Staudt published his "Geometrie der Lage," a mathematics utterly freed from any idea of quantity, a mathematics strictly qualitative, and De Morgan printed his fundamental treatise, called "Formal Logic ; or, the Calculus of Inference, Necessary and Probable."

The great memoirs produced in 1850, 1858, 1860, 1863 are preserved, if buried, in the inaccessible "Cambridge Philosophical Transactions." De Morgan's great combination of logical with mathematical learning, and his prominent position in London, the great metropolis, made him the man to whom resorted all the Circle-Squarers, Angle-Trisectors, Perpetual-Motionists, Triangle-Angle-sumers. Adding this curious experience to his great bibliographic knowledge of what had been attempted in that way in the past, he formed a large book called "A Budget of Paradoxes," which is one of the most interesting treatises ever written on what may be called extended fallacies. This charming book has steadily advanced in market price until I find it cited in Macmillan's Catalogue No. 245 (1086) at fifty shillings per copy.

It was De Morgan who first gave a thorough treatment of contrary, nega-

tive, or contradictory terms, though in just the sense I have heard babies use them while learning our language. But remember it is Darwinism that has since taught the world to learn at the feet of babes. Bain says : "According to the true view of contrariety, as given by De Morgan, the negative is a remainder, gained by the subtraction of the positive from the universe ; the negative of X is $U-X$, and may be symbolized by a distinct mark, x ; whence X and x are the opposites under a given universe ; not $-X$ is x, and not$-x$ is X."

Of the separation of logic and mathematic De Morgan says : "The effect has been unfortunate. . . . The sciences of which we speak may be considered either as disciplines of the mind, or as instruments in the investigation of nature and the advancement of the arts. In the former point of view their object is to strengthen the power of logical deduction by frequent examples ; to give a view of the difference between reasoning on probable premises and on certain ones by the construction of a body of results which in no case involve any of the uncertainty arising from the previous introduction of what may be false ; to establish confidence in abstract reasoning by the exhibition of processes whose results may be verified in many ways ; to help in enabling to acquire correct notions of generalization ; to give caution in receiving that which at first sight appears good reasoning ; to instill a correct estimate of the powers of the mind by pointing out the enormous extent of the consequences which may be developed out of a few of its most fundamental notions ; and to give the luxury of pursuing a study in which self-interest cannot lay down premises nor deduce conclusions.

As instruments in the investigation of nature and the advancement of the arts it is the object of these two sciences to find out truth in every matter in which nature is to be investigated, or her powers and those of the mind to be applied to the physical progress of the human race, or their advancement in the knowledge of the material creation."

De Morgan was fond of laughing at the metaphysicians. He says : "We know all about *can* and *cannot* from our cradles ; we never feel the same assurance about *is* and *is not*.

A philosopher, in a dark age, may determine to set out with a knowledge of the naturally possible and impossible ; but not even a philosopher ever pretended to set out with a knowledge of the existent and non-existent."

Aristotle and all the old logicians said that the whole of the middle term must be taken in at least one of the premises. As they put it, the middle term must be distributed at least once in the premises, otherwise the minor term may be compared with one part and the major with another part of it. From

Some men are poets,
Some men are Indians,

nothing follows. But the Aristotelians were wrong, as De Morgan clearly showed in his doctrine of Plurative Judgments. For example, if we have given the premises,

Most men are uneducated,
Most men are superstitious,

according to Aristotle we are not warranted in drawing any conclusion; for the middle term is men, and in neither premise is anything said about all men.

But, in point of fact, we can draw the perfectly valid conclusion,

Some uneducated men are superstitious.

Again Aristotle is contradicted by numerically definite judgments. In these there is inference when the quantities of the middle term *in the two premises together* exceed the whole quantity of that term.

Lambert first thought of this principle. De Morgan reconceived it and extended its use.

Suppose we grant the premises,

Two-thirds of all adults are women.

The number of women who have been married is never greater than the *total* number of men. It follows that half the entire number of women are single.

Still, easy and certain as such reasoning is, it may be difficult to a logician trained only in the traditional logic.

In a Princeton "Manual of Logic" the only numerically definite syllogism given was erroneous, and stood so for years. I stated this to the author, and in the latest stereotyped edition it has been changed. The Syllogism he gave was as follows:

"60 out of every 100 are unreflecting.
"60 out of every 100 are restless.
"Therefore, 20 out of every 100 restless persons are unreflecting."

After pointing out to him the fault in what he had been teaching for years, the following has been substituted:

"60 out of this 100 are unreflecting.
"60 out of this 100 are restless.
" ∴ 20 restless persons are unreflecting."

But De Morgan's greatest work was connected with his development of the Logic of Relatives, independently discovered by Robert Leslie Ellis after reading Boole's "Laws of Thought."

One of De Morgan's last memoirs, in the tenth volume of the "Cambridge Transactions," was on the Logic of Relations, which is, in the mathematical sense, a far-reaching generalization of the old logic. In our modern mathematics everything is generalized as far as possible. Every study of a generalization gives additional power over the particular. We need to go beyond and look back from an elevation.

Any first-rate mathematician working in logic would attempt to generalize, and, in fact, Boole generalized the scholastic logic in a manner entirely different from De Morgan. In De Morgan's view of the subject, the purely formal proposition with judgment wholly void of matter, is seen in "There is the probability x that X is in the relation L to Y." The syllogism is the determination of the relation which exists between two objects of thought by means of the relation

in which each of them stands to some third object which is the middle term. The pure general form of the syllogism, when its premises are absolutely asserted, is as follows: X is in the relation L to Y, Y is in the relation M to Z; therefore X is in the relation "L of M," compounded of L and M, to Z. In ordinary logic the actual composition of the relation is made by our consciousness of its *transitive* character. A relation is transitive when, being compounded with itself, it reproduces itself; that is, L is transitive when every L of L is L; for example "brother."

Thus De Morgan broke away from that paltry narrowness which asserts that our minds in pure thinking can use nothing but the relation of identity; from the Jevons sophism that thought cannot move because all thought is the substitution of identicals.

So we see that in logic, as in mathematics, we may develop a whole system of theorems about symbols which are to be used in a given manner; and then to make this whole system true of a desired relation or subject matter we have only to show that this relation or subject matter fulfills the few fundamental principles of the system.

De Morgan treated of convertible and inconvertible relatives, repeating relatives; non-repeating relatives, transitive and intransitive relatives, and inaugurated a general system.

To what tremendous estate this system has grown may be seen in the 649 pages of Ernst Schroeder's Treatise, "Algebra und Logik der Relative;" Leipzig, Teubner, 1895, on whose first page, as founder of the system, stands the name of Augustus De Morgan.

ON THE SOLUTION OF THE QUADRATIC EQUATION.

By G. A. MILLER, Ph. D., Paris, France.

One of the most important applications of substitution groups occurs in the theory of the solution of algebraic equations. It seems desirable that a fairly complete discussion of the solution of the quadratic equation should precede the study of this application. It is hoped that this discussion will not be without interest in itself even if the facts with which we have to deal are well known. As we shall need a clear idea of the *domain of rationality* we shall first develop several elementary concepts which involve the notion of groups and naturally lead to the more general concept of the domain of rationality.

Let us first consider the totality of numbers (T_1) formed by all the positive integers,* each positive integer occurring once and only once. By adding

*We shall throughout confine our attention to the finite. Not only are the numbers to be regarded as finite but the number of times that a given operation is to be performed is also to be considered finite.

any one of these to itself or to any other number in T_1 we obtain no new number. We may therefore say that T_1 forms a *group* (G_1) *with respect to addition.* It is evident that T_1 also forms a group (G_2) with respect to multiplication. Each of these two groups contains an indefinite number of subgroups.* To every subgroup of G_1 there corresponds a subgroup of G_2 which contains the same numbers but the converse is not true. For instance, all the positive integral powers of any prime number form a subgroup of G_2† but they do not form a subgroup of G_1.

We shall not enter upon the discussion of the subgroups found in G_1 and G_2 but only call attention to a few of the most simple ones. All the even positive integers clearly form a subgroup of both of these groups. We may inquire what is the smallest subgroup that contains any given positive integer a. In G_2 this subgroup is a^α, $\alpha=1, 2, 3, \ldots\ldots$ while in G_1 this subgroup is βa, $\beta=1, 2, 3, \ldots\ldots$ Hence unity is a subgroup of G_2 but not of G_1. As G_1 contains no subgroup that involves unity we may say that it is generated by this element. G_2 is generated by all the prime numbers together with unity.

The totality of negative integers (T_2 also forms a group with respect to addition but it does not form a group with respect to mnltiplication. It is clear that the smallest group with respect to multiplication that contains T_2 must also contain T_1. That is, $T_1+T_2=T_3$ is a totality which forms a group with respect to multiplication. T_3 also forms a group with respect to each of the operations addition and subtraction. The group with respect to subtraction does not contain any subgroup involving either T_1 or T_2, that with respect to multiplication contains a subgroup that involves T_1 but none that involves T_2, while that with respect to addition contains a subgroup that includes T_1 and also one that includes T_2.

Among the numbers which are now in common use those included in T_1 are perhaps of special importance as is also indicated by the fact that they are frequently called the natural numbers. In considering the groups which certain totalities of numbers form with respect to given operations it is therefore of special importance to inquire into the smallest groups that contain T_1. We have already seen that with respect to subtraction this smallest group includes other numbers than those contained in T_1. Similarly we observe that with respect to division‡ this smallest group includes an additional totality of numbers, viz: the fractions whose numerators and denominators are positive integers.

Instead of inquiring into the smallest totality of numbers that contains T_1 and forms a group with respect to a given operation we may also inquire into the smallest totality that contains T_1 and forms a group with respect to each of a given number of different operations. For instance, the smallest totality

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*The term subgroup is used, as usual, to represent a group that is contained in the larger group under consideration.

†For brevity we shall say that certain elements of a group form a subgroup instead of saying that these elements form a group if they are combined according to the same operation as the elements of the group.

‡As we have excluded the infinite we are not allowed to divide by 0. We may regard this as an impossible operation in the region to which we confine our observations.

(T_4) that contains T_1 and forms a group with respect to each of the four fundamental operations—addition, subtraction, multiplication, and division—is that which is formed by all the rational numbers. If we form the smallest totality that contains unity and forms a group with respect to each of these four operations we evidently arrive again at T_4. On this account the totality of all the rational numbers is generally called the domain unity. This is the simplest domain of rationality.*

It would be of interest to consider all the different types of subgroups of the groups formed by T_4 with respect to the given fundamental operations. We shall not enter into this field as the matter has probably been sufficiently developed for our present purposes. We would remark, however, that a careful study of these matters seems to us to be one of the simplest roads towards forming a clear notion of groups as well as of the domain of rationality.

The operations which we have thus far considered may be represented by the simple equations:

$$a+b=x, \quad a-b=x, \quad a\times b=x, \quad a\div b=x.$$

We have seen that x is always a rational number when both a and b are such numbers. In other words, we have seen that when we take a and b from the totality of numbers represented by T_4, x will also belong to this totality, but when we take a and b from one of the other totalities of numbers that have been considered—T_1, T_2, T_3—we cannot affirm that x belongs to the same totality in all the equations.

At an early stage in the development of mathematics it became necessary to solve equations of a higher than the first degree. One of the great difficulties which presented itself at this point was the fact that T_4 does not form a group with respect to the algebraic operations whose degree exceeds one. As long as these operations can be represented by a binomial equation of the form

$$x^n=a$$

n being a positive integer and a being a positive rational number, the difficulty was not much greater than that which had to be overcome at preceding stages, for the extension of the totality of numbers in such a way as to include irrational numbers does not seem an irrational adventure, even if the association of these numbers with matters of observation is not so direct and evident as it had been in the preceding cases.

*The totality of rational functions with integral coefficients of a given number of determinate or indeterminate independent quantities (R, R', R'', ...) is called, after Kronecker, a domain of rationality. In other words, it is the smallest totality that contains these quantities and forms a group with respect to the four fundamental operations. In the domain unity we have clearly only one such quantity and this is determinate, viz: $R=1$. We shall soon consider a domain of one indeterminate quantity or parameter. Sometimes the given R's are defined as indeterminate parameters. According to this definition the domain unity contains no parameter. This domain is clearly generated by any rational finite number except 0. It may therefore be called the domain 2, 3, ... as well as the domain 1.

A much more serious difficulty presented itself when it was required to introduce the operation indicated by the general quadratic equations

$$x^2 + bx + c = 0. \qquad (A).$$

Even if we take b and c from the totality T_1 it often happens that we can not find any number among those that have been considered which comes near towards satisfying this equation. Hence the totality of real numbers (T_5) can clearly not form a group with respect to this operation. With respect to the four fundamental operations T_5 forms a group which contains T_4 as a subgroup.

It should however not be inferred that none of the numbers that have been considered form a group with respect to the operation (A). In order that a single number (α) may have this property it is necessary and sufficient that $x = \alpha$ satisfies the equation

$$x^2 + \alpha x + \alpha = 0.$$

Hence there are two numbers ($-\frac{1}{2}$ and 0) each of which forms a group with respect to the quadratic equation. 0 also forms a group with respect to the four fundamental operations but $-\frac{1}{2}$ does not have this property. It should also be observed that when $\alpha = -\frac{1}{2}$ it is not the only value of x that satisfies the given equation and that the term group has therefore to used in a somewhat restricted sense when we say that $-\frac{1}{2}$ forms a group with respect to the quadratic equation.

But, even if it was known that certain special numbers form a group with respect to the operation (A), and, what is more important, that x belongs to the totality of numbers T_5 for a large number of types of (A) yet the matter remained in an unsatisfactory state as long as no totality of numbers was known which includes T_5 and forms a group with respect to (A). The struggle for light on this point was a long one, reaching far into our century. We cannot enter into a history of this struggle. It must suffice to state that the triumph was largely due to the elegant geometrical representation of the complex number by means of points in the plane. This was not the first nor last assistance that algebra has received from his sister geometry. On the other hand, algebra has a very brilliant record of services rendered to his ambitious sister.

The importance of the adoption of the complex numbers (T) cannot be fully appreciated if we confine our attention to the quadratic equation. If the general algebraic operations of each of the following degrees had again required equally great extension in the number system this would soon have become exceedingly difficult and the progress in the solution of the algebraic equations could not have been so rapid. The great importance of the adoption of the totality of numbers T may therefore be said to be due to the fact that it forms a group with respect to the general algebraic operation indicated by the following equation

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots + a_n = 0 \qquad (B),$$

where a_0, a_1, a_2, ..., a_n belong to T and n belongs to T_1. In other words if the coefficients of (B) belong to the domain of the indeterminate parameter

$$R=\sqrt{a} \qquad (a \text{ belonging to } T_5)$$

x will also belong to this domain. Or again, B can be resolved into factors without using numbers that lie outside of T.

While we cannot fully appreciate the importance of the adoption of the complex numbers when we confine our attention to the quadratic equation, yet in this equation we see the source of T and with it we naturally associate the wonderful progress achieved by means of T. It is however not our object to enter upon the discussion of the important position which the quadratic equation occupies in the development of mathematics even if an idea of this position naturally increases the interest in this equation and hence also in its discussion as an algebraic operation.

To convey an idea of the importance of the concept domain of rationality we shall consider an application. Suppose that we have an equation of the form (B) and that its coefficients belong to a certain domain of rationality T' while none of its factors belong to this domain.* Suppose that we have any other equation whose coefficients also belong to T' and that these two equations have a common root. We can then say that the first equation is a factor of the second. In other words, we know that all the roots of the first equation are also roots of the second. The truth of this statement follows directly from the fact that the coefficients of the greatest common divisor of the two equations must belong to T'. This greatest common divisor must therefore be the first member of the first equation.

If we have any complex number

$$a+bi$$

the quadratic equation which contains this number and its conjugate for its roots is

$$x^2-2ax+a^2+b^2=0. \qquad (C).$$

The coefficients of this equation belong to T_5. Suppose now that we have any other equation that contains $a+bi$. From the proof just given it follows that it must also contain (C). The same remarks clearly apply to surd roots of the form

$$a\pm\sqrt{b}.$$

Hence we see that the given statement includes the theorems that if an equation with real coefficients contains the root $a+bi$ it also contains the root $a-bi$, and if an equation whose coefficients are rational contains a surd root of the form $a+\sqrt{b}$ it also contains $a-\sqrt{b}$ as a root.

*In this case we say that the equation is irreducible in the domain T'.

THE NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M. (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from May Number.]

SCHOLION.

Nevertheless it might be doubted, whether, from whatever point K (assumed indeed in BX before the meeting of this BX with the other AX) erected toward the parts of the straight AX, a perpendicular must meet this (fig. 29) in some point L; provided of course those two, before the aforesaid meeting, are assumed ever more to approach each other mutually [and not to meet at any finite remove].

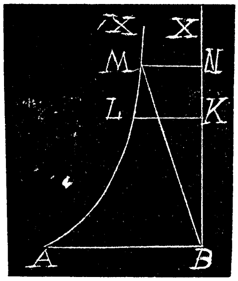


Fig. 29.

But I say it will follow completely thus.

Proof. Let there be assigned in BX any point whatever K. In AX is taken a certain AM equal to the sum of this BK and of twice AB.

Then from the point M is drawn to BX (according to Eu. I. 12) the perpendicular MN. According to the present supposition, MN will be less than AB. Wherefore AM (made equal to the sum of BK and of double AB) will be greater than the sum of BK, AB, and NM. Now it behooves to show this same AM to be less than the sum of BN, AB, and MN, that thence it may follow this BN is greater than the aforesaid BK, and therefore the point K lies between the points B and N.

Join BM. The side AM will be (from Eu. I. 20) less than the two remaining sides together AB and BM. Again the side BM (from the same Eu. I. 20) will be less than the two sides together BN and MN. Therefore the side AM will be by much less than the three sides together AB, BN, and NM. But this was to be shown, in order to deduce that the point K lies between the points B and N. Thence however it follows, that the perpendicular from the point K erected toward the parts of AX must meet this in some point L stationed between the points A and M; else obviously (against Eu. I. 17) it must cut either AB or MN perpendiculars to BX.

Quod etc.

[To be Continued.]

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJ. F. YANNEY, A.M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from December Number.]

XXII. Let ABC be a triangle, right-angled at C. Assume $CB<CA$. Complete the figure as indicated, the point E being at the middle of AB.

There are three cases: (1) when E is without the circle, (2) in the circumference, (3) within the circle. We treat but one case, when E is within the circle. The other cases may be treated in a similar manner.

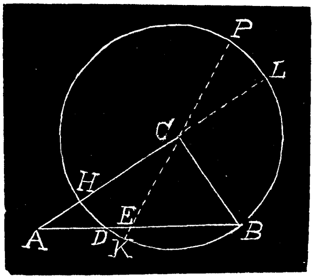


Fig. 17.

$$AD=\frac{AH\cdot AL}{AB}=\frac{b^2-a^2}{c}\ \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$DE=\frac{KE\cdot EP}{EB}=\frac{(a-\frac{1}{2}c)(a+\frac{1}{2}c)}{\frac{1}{2}c}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Add (1) and (2), $\frac{1}{2}c=\dfrac{a^2+b^2-\frac{1}{2}c^2}{c}$. $\therefore\ c^2=a^2+b^2$.

NOTE. This method is credited, by Wipper, to Krueger, 1746. Though in the original, the legs are arbitrarily assumed as unequal, we may pass, by the theory of limits, to the case when the legs are equal.

XXIII. Let ABC be a triangle, right-angled at C. Circumscribe the triangle by a circle. Complete the rectangle. Then,

$$AB\cdot CD=CB\cdot AD+AC\cdot DB,\ \text{or}\ c^2=a^2+b^2.$$

NOTE. Thought this method must have been known independently to many persons, it seems that its first appearance in print was in 1859, in Runkle's *Mathematical Monthly*.

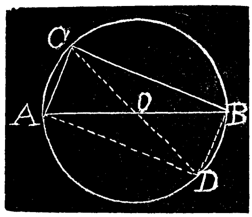


Fig. 18.

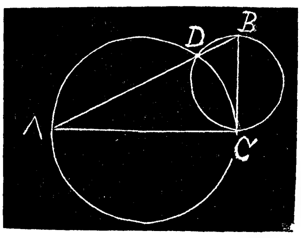


Fig. 19.

XXIV. Let ABC be a triangle, right-angled at C. Construct circles with AC and BC as diameters, respectively. These circles will intersect in AB, as at D.

Then, $\overline{AC}^2=AD\cdot AB$, and $\overline{BC}^2=BD\cdot AB$.

Add, $\overline{AC}^2+\overline{BC}^2=AB(AD+BD)=\overline{AB}^2$.

NOTE. This is one of Richardson's.

XXV. Let ABC be a triangle, right-angled at C. Complete the figure as indicated.

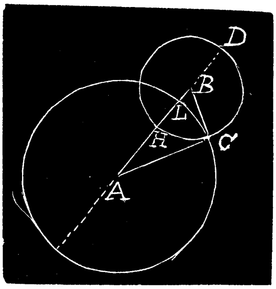


Fig. 20.

Then, $\overline{AC}^2=AH\cdot AD$, and $\overline{BC}^2=BL\cdot BE$.

Add, $\overline{AC}^2+BC=AH\cdot AD+BL\cdot BE$

$=(AB-BC)(AB+BC)+(AB-AC)(AB+AC)$.

$\therefore\ \overline{AB}^2=\overline{AC}^2+\overline{BC}^2$.

Richardson's method is somewhat different. Thus:

$$\overline{AC}^2+\overline{BC}^2=AH(AB+BC)+BL(AB+AC)$$

$$=AB(AH+BL)+AH\cdot BC+BL\cdot AC+BL\cdot AB-BL\cdot AB$$

$$=AB(AH+BL)+AH\cdot BC+BL(BL+2AC)-BL\cdot AB$$

$$=AB(AH+BL)+AH\cdot BH+\overline{BH}^2-BL(AH+BH)$$

$$=AB(AH+BL)+(AH+BH)(BH-BL)$$

$$=AB(AH+BL)+AB\cdot HL=AB(AH+BL+HL)=\overline{AB}^2.$$

XXVI. Fig. 20.

$EH : ED :: HL : DL$. (See Olney, §971). $\therefore\ EH\cdot DL=ED\cdot HL$,

or $(AC+AB-BC)(BC+AB-AC)=(AC+AB+BC)(AC+BC-AB)$.

$\therefore\ \overline{AB}^2=\overline{AC}^2+\overline{BC}^2$.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield; Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

70. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

A owes me \$100 due in 2 years, and I owe him \$200 due in 4 years; when can I pay him \$100 to settle the account equitably, money being worth 6% ?

I. Solution by the PROPOSER, and P. S. BERG, Larimore, North Dakota.

Let x=the time.

Now the amount of \$200 for $(x-4)$ years—the amount of \$100 for $(x-2)$ years must=\$100.

$200+12(x-4)=152+12x$=amount of \$200 for $(x-4)$ years at 6%.

$100+6(x-2)=88+6x$=amount of \$100 for $(x-2)$ years at 6%.

$\therefore\ (152+12x)-(88+6x)=100. \quad \therefore\ x=6$ years.

II. Solution by Hon. J. H. DRUMMOND, LL. D., Portland, Maine.

Computing at simple interest,

$\$\frac{200}{124}-\$\frac{100}{112}=\$72\frac{1}{217}$, amount equitably due now.

Hence, $\$100-\$72\frac{1}{217}=\$27\frac{216}{217}$ to be earned as interest.

$\$27\frac{216}{217}\div(72\frac{1}{217}\times\frac{6}{100})=5\frac{31}{75}$, the number of years required.

III. Solution by NELSON S. RORAY, South Jersey Institute, Bridgeton, New Jersey.

In this I assume interest to remain unpaid until the time of settlement, and to *draw no interest.*

A owes me to-day the present worth of \$100 due in 2 years at 6%, or \$89.29.

I owe A to-day the present worth of \$200 due in four years at 6%, or \$161.29.

That is, I owe A \$72 more than he owes me. Hence the problem reduces itself to the question, when will the excess of my interest over his plus \$72 amount to \$100? That is, my interest must exceed his by \$28.

My yearly excess is \$4.32. Hence to gain \$28, 6.481 years will be required.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

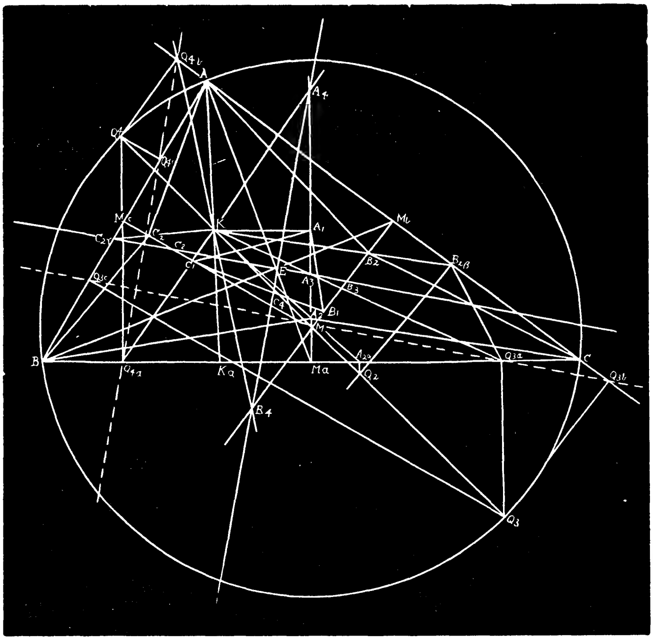
SOLUTIONS OF PROBLEMS.

65. Proposed by I. J. SCHWATT, Ph. D., Professsor of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania.

The axes of the ellipse isogonal to Lemoine's line with respect to a triangle (Steiner's ellipse), are parallel to Simson's lines belonging to the extremities of Brocard's Diameter.

Solution by F. M. McGAW, A. M., Professor of Mathematics, Bordentown Military Institute, Bordentown, New Jersey.

Let the triangle be ABC ; center of circumcircle, M ; $A_1B_1C_1$, the Brocard triangle with vertices on parallels thro' Grebe's point and perpendiculars at mid-points of sides of ABC. It is known that ABC and $A_1B_1C_1$ are similar. Let E be the medio-centre, or centre of gravity, of ABC. It is known that E is also medio-centre of $A_1B_1C_1$. Let MK, Brocard's diameter (or the diameter of circle about Brocard's triangle) be produced to meet circumcircle of ABC in the points Q_3 and Q_4.



By the construction, the vertices of Brocard's triangle are also the vertices of three similar isosceles triangles ; for these isosceles triangles have as altitudes the perpendiculars from Grebe's point upon the sides of ABC, and it is known that these perpendiculars are as the sides. Hence the triangles have bases and altitudes proportional, and therefore are similar.

If, now, any three similar isosceles triangles be constructed upon the sides of ABC, their vertices A_2, B_2, C_2, will be the vertices of a triangle having the same medio-centre as ABC or $A_1B_1C_1$. The proof of this is similar to that which is known to establish E the same for ABC, and $A_1B_1C_1$.

Draw KA_2, KB_2, KC_2 to meet sides of triangle ABC in points $A_{2\alpha}$, $B_{2\beta}$, $C_{2\gamma}$ respectively. Then triangle $A_{2\alpha}B_{2\beta}C_{2\gamma}$ is similar to the triangle $A_2B_2C_2$, and centre of similitude is K. This may be proved as follows : Erect a perpendicular at $A_{2\alpha}$ to cut Brocard's diameter (produced) at Q_2, then

$$A_2M_a : KK_a = A_{2\alpha}A_2 : A_{2\alpha}K = Q_2M : Q_2K. \qquad (1).$$

Triangles A_2BC, and B_2AC are similar by construction, hence

$$A_2M_a : B_2M_b = M_aC : M_bC = a : b = A_1M_a : B_1M_b,$$

where $a : b$ is ratio of two sides of triangle ABC.

We may write this last

$$A_2M_a : A_1M_a = B_2M_b : B_1M_b = B_{2\beta}B_2 : B_{2\beta}K = Q_2M : Q_2K \qquad (2).$$

From (1) and (2) we have,

$$A_{2\alpha}A_2 : A_{2\alpha}K = B_{2\beta}B_2 : B_{2\beta}K,$$

hence the lines A_2B_2 and $A_{2\alpha}B_{2\beta}$ become parallel, and if the same course of reasoning be pursued with regard to the other sides, the triangles are seen to be similar, with K the center of similitude.

Now, from the equality, $B_{2\beta}B_2 : B_{2\beta}K = Q_2M : Q_2K$, it follows that $B_{2\beta}Q_2$ is parallel to B_2M; and since B_2M is perpendicular to AC, $B_{2\beta}Q_2$ is also perpendicular to AC. Similarly, the perpendicular to AB at $C_{2\gamma}$ passes through Q_2, and we have already that the perpendicular to BC at $A_{2\alpha}$ passes through Q_2. If Q_2 be caused to coincide with either Q_3 or Q_4, then triangle $A_{2\alpha}B_{2\beta}C_{2\gamma}$ will degenerate into the straight lines $Q_{3a}Q_{3b}Q_{3c}$, and $Q_{4a}Q_{4b}Q_{4c}$ which are the Simson lines belonging to Q_3 and Q_4. Also triangle $A_2B_2C_2$ will degenerate into the straight lines $A_3B_3C_3$, and $A_4B_4C_4$, which are parallel to Simson's lines, to Q_3, Q_4. These lines will also pass through the medio-centre, E, since the triangles which degenerate continually have E as the medio-centre. Since the Simson lines are perpendicular to each other (see Geometry of Simson lines), these last mentioned lines through E, are perpendicular to each other. Since we know that the ellipse (Steiner's) has these lines for axes, the proposition is proved. Q. E. D.

NOTE. The above solution I got from Dr. Schwatt. An elegant demonstration of properties of the ellipse is given in Schwatt's Isogonal Curves, (Leach, Shewell & Sanborn, New York). F. M. M.

This problem was also solved by Prof. G. B. M. Zerr. Prof. William Hoover did not solve it, but referred to the proof given in *Casey's Analytical Geometry*, Edition of 1893, Articles 364, 365 (Cor. 1).

66. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

The locus of points whose polars with respect to a given parabola touch the circle of curvature at the vertex is an equilateral hyperbola.

I. Solution by the PROPOSER.

By Salmon's *Conic Sections*, Sixth Edition, Ex. 4, page 234, the equation to the circle osculating a parabola $y^2=px$......(1) at (x', y') is

$$(p^2+4px')(y^2-px)=\{2yy'-p(x+x')\}\{2yy'+px-3px'\}\ldots\ldots\ \ldots(2).$$

At the vertex, $x'=0$, $y'=0$, and (2) becomes

$$x^2+y^2-px=0 \qquad \text{(3).}$$

If (x_1, y_1) be any point on the required locus, its polar with respect to (1) is

$$px-2y_1y+px_1=0 \qquad \text{(4).}$$

The condition that (4) touches (3) is

$$x_1{}^2-y_1{}^2-px_1=0 \qquad \text{(5),}$$

an equilateral hyperbola.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland; CHAS. C. PURYEAR, Professor of Mathematics, Agricultural and Mechanical College, College Station, Texas; and G. B. M. ZERR, Texarkana, Ark.

Let $y^2=4ax$ be the equation to the parabola, (b, c) any point.

Then $cy=2a(x+b)$......(1) is the polar of (b, c). $x^2+y^2=ax$.......(2) is the circle of curvature at the vertex. The value of y from (1) in (2) gives

$$c^2x^2+4a^2x^2+8a^2bx+4a^2b^2=ac^2x \qquad \text{(3).}$$

From (3) we find the condition that (1) should be tangent to (2) to be

$$a^2c^4=8a^3bc^2+16a^2b^2c^2. \quad \therefore c^2=8ab+16b^2.$$

$$\therefore a^2=(4b+a)^2-c^2 \qquad \text{(4).}$$

(4) represents an equilateral hyperbola.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

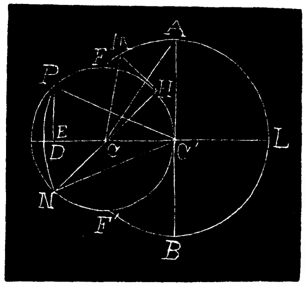
55. Proposed by GEORGE LILLEY, Ph. D., LL. D., Principal of Park School, 394 Hall Street, Portland, Oregon.

A horse is tethered by a rope, a feet long, fastened to a post in a circular fence enclosing a circular piece of grouhd b feet in diameter. If the horse is tethered outside of the fence over how much ground can he feed? If he is inside the fence over how much ground can he feed? b is greater than a in each case.

I. Solution by the PROPOSER.

Consider area $C'HFMA$. $C'A=C'P=a$, feet. $CC'=CH=\frac{1}{2}b$, feet. Let $\angle HCF=\phi$. $HM=$arc $HF=\frac{1}{2}b\phi$. The area element is $\frac{1}{2}(HM)^2d\phi$.

$$\therefore \text{ area } C'HFMA=\tfrac{1}{8}\int_0^{\frac{2a}{b}} b^2\phi^2 d\phi, =\frac{a^3}{3b}.$$



Hence, when the horse is outside, he can graze over

$$\frac{a^2}{6b}(4a+3b\pi), \text{ square feet.}$$

Draw PE at right angles to DC'. Let $x=CE$, and $\angle PC'E=\theta$.

$$\text{Area of sector } PC'D=\int_0^\theta\int_0^a da d\theta, =\tfrac{1}{2}a^2\cos^{-1}\frac{b+2x}{2a}, =\text{sector } NC'D.$$

$$\text{Area of sector } PCC'HF=\tfrac{1}{8}b^2\times\text{angle } PCC', =\tfrac{1}{8}b^2\cos^{-1}\left(-\frac{2x}{b}\right).$$

$$\text{Area of triangle } PCC'=\tfrac{1}{8}b\sqrt{b^2-4x^2}.$$

$$\therefore \text{ area of segment } PFHC'=\tfrac{1}{8}b^2\cos^{-1}\left(-\frac{2x}{b}\right)-\tfrac{1}{8}b\sqrt{b^2-4x^2}, =\text{segment } HC'F'.$$

$$x=\frac{2a^2-b^2}{2b}.$$

Hence, area of $C'HPNF'=a^2\cos^{-1}\frac{a}{b}+\frac{1}{4}b^2\cos^{-1}\frac{b^2-2a^2}{b^2}-\frac{1}{2}a\sqrt{b^2-a^2}$, square feet ; the space grazed over inside the fence.

II. Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

In first part required area equals that of semicircle $ALB+2\times AMFHC$. Let HM be a portion of rope unwound and equal to HF. Let $\angle HCF=\phi$, $\angle MCF=\theta$, $CM=\rho$, and take CF for polar axis. Then

$$\rho^2=\tfrac{1}{4}b^2+\overline{HM}^2, =\tfrac{1}{4}b^2+\tfrac{1}{4}(b^2\phi^2).$$

$$\phi=\frac{2}{b}\sqrt{4\rho^2-b^2}. \qquad \frac{b}{2\rho}=\cos(\phi-\theta), =\cos\left(\frac{2}{b}\sqrt{4\rho^2-b^2}-\theta\right).$$

$$\theta=\frac{2}{b}\sqrt{4\rho^2-b^2}-\cos^{-1}\left(\frac{b}{2\rho}\right).$$

$$\frac{d\theta}{d\rho}=\frac{4\rho}{b\sqrt{4\rho^2-b^2}}-\frac{b}{2\rho^2\sqrt{1-\dfrac{b^2}{4\rho^2}}},=\frac{\sqrt{4\rho^2-b^2}}{b\rho}.$$

$$\frac{dA}{d\rho}=\frac{dA}{d\theta}\frac{d\theta}{d\rho},=\frac{\rho^2}{2}\times\frac{\sqrt{4\rho^2-b^2}}{b\rho},=\frac{\rho\sqrt{4\rho^2-b^2}}{2b}$$

Limits of ρ for $CFMA$ are seen to be $\frac{1}{2}b$ and $\sqrt{a^2+\frac{1}{4}b^2}$.

$$\therefore \text{ Area } CFMA=\frac{1}{2b}\int_{\frac{1}{2}b}^{\sqrt{a^2+\frac{1}{4}b^2}}\rho\sqrt{4\rho^2-b^2}\,d\rho,=\frac{a^3}{3b}.$$

Area $FHCAM=CFMA+C'CA-CFHC',=a^3/3b+ab/4-ab/4=a^3/3b$.

$\therefore$ Area $F'BLAFHC'=2a^3/3b+\pi a^2/2$.

Internal area is composed of $2\times$segment $PHC'+$sector $PDNC'$

$$\sin\tfrac{1}{2}\angle PCC'=(\tfrac{1}{2}a/\tfrac{1}{2}b)=a/b.\quad \angle PCC'=2\sin^{-1}(a/b).$$

$$\angle PCN=2\pi-4\sin^{-1}(a/b).\quad \angle PC'N=\pi-2\sin^{-1}(a/b).$$

$$\text{Sec. } PDNC'=\frac{a^2}{2}\left(\pi-2\sin^{-1}\frac{a}{b}\right),\ \text{sector } CPHC'=\frac{b^2}{8}\left(2\sin^{-1}\frac{a}{b}\right),=\frac{b^2}{4}\sin^{-1}\left(\frac{a}{b}\right)$$

$$\triangle PCC'=\tfrac{1}{2}a\sqrt{\tfrac{1}{4}b^2-\tfrac{1}{4}a^2},=\tfrac{1}{4}a\sqrt{b^2-a^2}.$$

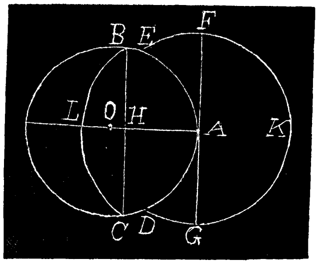
$$\text{Segment } PHC'=\frac{b^2}{4}\sin^{-1}\frac{a}{b}-\tfrac{1}{4}a\sqrt{b^2-a^2}.$$

$$\therefore \text{ Internal area }=\frac{a^2}{2}\left(\pi-2\sin^{-1}\frac{a}{b}\right)+\frac{b^2}{2}\sin^{-1}\frac{a}{b}-\tfrac{1}{2}a\sqrt{b^2-a^2},$$

$$=\tfrac{1}{2}\pi a^2-\frac{a}{2}\sqrt{b^2-a^2}+(\tfrac{1}{2}b^2-a^2)\sin^{-1}\left(\frac{a}{b}\right).$$

III. Solution by G. B. M. YERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let A be the point where the horse is tethered. $AF=a$, $AO=b/2$. Area $EADGKFE=2$ area $EAF+$area of semicircle GKF.



$$\therefore A=\int\rho^2 d\theta+\tfrac{1}{2}\pi a^2\ ;\ \text{but } \rho=\tfrac{1}{2}b\theta.$$

$$\therefore\ A=\tfrac{1}{4}b^2\int_0^{2a/b}\theta^2 d\theta+\tfrac{1}{2}\pi a^2,=\frac{a^2}{6b}(4a+3\pi b).$$

Let $x^2+y^2=\frac{1}{4}b^2$, be the equation to circle center O. $(x-\frac{1}{2}b)^2+y^2=a^2$, be the equation to circle center A.

$$\therefore\ OH=\frac{b^2-2a^2}{2b},\quad \therefore\ BH=\frac{a}{b}\sqrt{b^2-a^2}.$$

A'=area of segment BLC+area of segment BAC,

$$=\frac{b^2}{4}\left\{\sin^{-1}\left(\frac{2a}{b^2}\sqrt{b^2-a^2}\right)-\frac{2a(b^2-2a^2)\sqrt{b^2-a^2}}{b^4}\right\}$$

$$+a^2\left\{\sin^{-1}\frac{\sqrt{b^2-a^2}}{b}-\frac{a\sqrt{b^2-a^2}}{b^2}\right\},$$

$$=\tfrac{1}{4}b^2\cos^{-1}\left(\frac{b^2-2a^2}{b^2}\right)+a^2\cos^{-1}\frac{a}{b}-\frac{a}{2}\sqrt{b^2-a^2}.$$

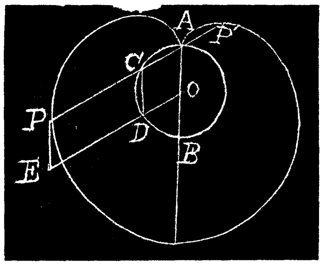
Also solved by *J. SCHEFFER* and *A. H. HOLMES.*

56. Proposed by B. F. BURLESON, Oneida Castle, New York.

Find (1) the length s of the closed curve of the cardioid; (2) its area A; (3) if made to revolve about its axis $2a$, find the maximum longitudinal circumference C of the solid generated; (4) find the surface K of the same; (5) its volume V; (6) the distance x_0 of the center of gravity of the solid from the origin O; and (7) the distance y_0 of the center of gravity of the plane curve from the origin O.

I. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland, and the PROPOSER.

Let $AB=a$ be the diameter of a circle. From A draw any chord AC. Make CP and $CP'=b$, then will the locus of P or P' be the Limaçon. If $AP=r$, $\angle PAB=\theta$, we find at once the polar equation of the Limaçon to be $r=a\cos\theta+b$. If $b>a$, the curve consists of but one loop; if $b<a$, it has two loops, and if $b=a$ the curve becomes the Cardioid, the polar equation of which is $r=a(1+\cos\theta)$. It can easily be shown that the cardioid is an epicycloid, the generating circle of which is equal to the fixed one; also, drawing through the center O of the circle a line parallel to AP cutting the circumference of the circle at D, and drawing through P a line parallel to CD, this line is a tangent to the cardioid at P. The different problems proposed are best solved by means of the polar equation of the curve.



$$(1).\quad \text{The length } s=2\int_0^\pi d\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2},=2a\int_0^\pi\cos\tfrac{1}{2}\theta d\theta,=8a.$$

$$(2).\quad \text{The area } A=\int_0^\pi r^2d\theta,=a^2\int_0^\pi(1+\cos\theta)^2d\theta,=\tfrac{3}{2}\pi a^2.$$

(3). To find the maximum ordinate, $r\sin\theta = a(\sin\theta + \frac{1}{2}\sin 2\theta$ is to be a maximum. By differentiation we find $\theta = 60^\circ$. $\therefore$ maximum ordinate $= \frac{3}{4}a\sqrt{3}$, and circumference $C = \frac{3}{2}\pi a\sqrt{3}$.

(4). Surface $K = 2\pi\int_0^\pi r\sin\theta d\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)}$,

$$= 4\pi a^2\int_0^\pi (1+\cos\theta)\sin\theta\cos\tfrac{1}{2}\theta d\theta, = -16\pi a^2\int_0^\pi \cos^4\tfrac{1}{2}\theta d\cos\tfrac{1}{2}\theta, = \frac{32\pi a^2}{5}.$$

For the distance x_0 of the center of gravity of this surface we have

$$Kx_0 = 2\pi\int_0^\pi r^2\sin\theta\cos\theta(ds/d\theta).d\theta, = 4\pi a^3\int_0^\pi (1+\cos\theta)^2\sin\theta\cos\theta.\cos\tfrac{1}{2}\theta d\theta,$$

$$= -64\pi a^3\left[2\int_0^\pi \cos^8\tfrac{1}{2}\theta d\cos\tfrac{1}{2}\theta.\int_0^\pi \cos^6\tfrac{1}{2}\theta d\cos\tfrac{1}{2}\theta\right], = \frac{320\pi a^3}{63};$$

$$\therefore x_0 = \frac{320\pi a^3}{63} \div \frac{32\pi a^2}{5} = \tfrac{50}{63}a.$$

(5). Volume, $V = 2\pi\int_0^\pi\int_0^{a(1+\cos\theta)} rdrd\theta.r\sin\theta, = \frac{2\pi a^3}{3}\int_0^\pi (1+\cos\theta)^3\sin\theta d\theta,$

$$= -\frac{2\pi a^3}{3}\int_0^\pi (1+\cos\theta)^3 d(1+\cos\theta), = \frac{8\pi a^3}{3}.$$

(6). The distance x_0 of this volume from the origin we find from

$$Vx_0 = 2\pi\int_0^\pi\int_0^{a(1-\cos\theta)} r^2drd\theta\sin\theta r\sin\theta, = \frac{\pi a^4}{2}\int_0^\pi (1+\cos\theta)^4\sin^2\theta d\theta,$$

$$= 32\pi a^4\int_0^\pi \cos^{10}\tfrac{1}{2}\theta\sin^2\tfrac{1}{2}\theta d\theta, = \frac{64\pi a^4\Gamma(\frac{3}{2})\Gamma(\frac{11}{2})}{2\Gamma(7)}, = \tfrac{21}{32}\pi^2a^4;$$

$$\therefore x_0 = \frac{21\pi^2a^4}{32} \div \frac{8\pi a^3}{3} = \tfrac{63}{256}\pi a.$$

(7). The distance x_0 of the center of gravity of the arc of the curve from the origin is found by

$$sx_0 = 2\int_0^\pi r\sin\theta.2a\cos\tfrac{1}{2}\theta.d\theta, = 16a^2\int_0^\pi \cos^4\tfrac{1}{2}\theta.\sin\tfrac{1}{2}\theta.d\theta,$$

$$= -32a^2\int_0^\pi \cos^4\tfrac{1}{2}\theta d(\cos\tfrac{1}{2}\theta), = \frac{32a^2}{5}, \quad \therefore x_0 = \frac{32a^2}{5} \div 8a = \tfrac{4}{5}a;$$

and the distance x'_0 of the area of the curve from the origin is found by

$$Ax'_0=\tfrac{2}{3}\int_0^\pi r^3\cos\theta d\theta,=\tfrac{2}{3}a^3\int_0^\pi(1+\cos\theta)^3\cos\theta d\theta,=\tfrac{16}{3}a^2\int_0^\pi(2\cos^8\tfrac{1}{2}\theta-\cos^6\tfrac{1}{2}\theta)d\theta,$$

$$=\tfrac{5}{4}\pi a^3. \qquad \therefore\ x'_0=\tfrac{5}{4}\pi a^3\div\frac{3\pi a^2}{2}=\tfrac{5}{6}a.$$

II. **Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.**

Let $r=a(1+\cos\theta)$, be the equation to the Cardioid.

(1). $s/2=2a\int_0^\pi\cos(\tfrac{1}{2}\theta)d\theta,=4a. \quad \therefore\ s=8a.$

(2). $A=2a^2\int_0^{2\pi}\cos^4(\tfrac{1}{2}\theta)d\theta,=\tfrac{1}{2}(3\pi a^2).$

(3). $C=2\pi\rho$, where $\rho=r\sin\theta,=a\sin\theta(1+\cos\theta)$. $d\rho=2a\cos^2\theta+a\cos\theta-a$.

$\therefore\ \cos\theta=\tfrac{1}{2}$ or -1. $\therefore$ for a maximum $\theta=60^\circ$.

$\therefore\ C=2\pi a(1+\cos\tfrac{1}{3}\pi)\sin\tfrac{1}{3}\pi,=\tfrac{1}{2}(3\sqrt{3}\pi a)$.

(4). $K=8\pi a^2\int_0^\pi\cos^3(\tfrac{1}{2}\theta)\sin\theta d\theta,=\tfrac{1}{5}(32\pi a^2).$

(5). $V=2\pi\int_0^\pi\int_0^{a(1+\cos\theta)}r^2\sin\theta d\theta dr,=\dfrac{16\pi a^3}{3}\int_0^\pi\cos^6(\tfrac{1}{2}\theta)\sin\theta d\theta.$

$\therefore\ V=\tfrac{1}{3}(8\pi a^3).$

(6). $$x_0=\frac{\int_0^\pi\int_0^{a(1+\cos\theta)}r^3\sin\theta\cos\theta d\theta dr}{\int_0^\pi\int_0^{a(1+\cos\theta)}r^2\sin\theta d\theta dr},=\tfrac{3}{2}a\frac{\int_0^\pi\cos^8(\tfrac{1}{2}\theta)\cos\theta\sin\theta d\theta}{\int_0^\pi\cos^6(\tfrac{1}{2}\theta)\sin\theta d\theta}$$

$\therefore\ x_0=\tfrac{4}{5}a.$

(7). $$g_0=\frac{\int_0^{2\pi}\int_0^{a(1+\cos\theta)}r^2\cos\theta d\theta dr}{\int_0^{2\pi}\int_0^{a(1+\cos\theta)}r d\theta dr},=\tfrac{4}{3}a\frac{\int_0^{2\pi}\cos^6(\tfrac{1}{2}\theta)\cos\theta d\theta}{\int_0^{2\pi}\cos^4(\tfrac{1}{2}\theta)d\theta}$$

$\therefore\ g_0=\tfrac{5}{6}a.$

Also solved by *C. W. M. BLACK*.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

40. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find the law of the force, in order that the orbit may be a Cassinian Oval.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let $r^4+2c^2r^2\cos 2\theta=m^4-c^4=a^4$ ......(1) be the equation to the oval.
Then $a^4u^4=2c^4u^2\cos 2\theta+1$, where $u=1/r$.

$$\frac{du}{d\theta}=\frac{c^2u\sin 2\theta}{c^2\cos 2\theta-a^4u^2}\text{ ....} (2).$$

$$\frac{d^2u}{d\theta^2}=\left\{\frac{\left(c^2\sin 2\theta\frac{du}{d\theta}+2c^2u\cos 2\theta\right)(c^2\cos 2\theta-a^4u^2)+\left(2c^2\sin 2\theta+2a^4u\frac{du}{d\theta}\right)c^2u\sin 2\theta}{(c^2\cos 2\theta-a^4u^2)^2}\right\}$$

$$=\left\{\frac{3c^4r^8+10r^4c^4m^4-3r^4c^8-(m^4-c^4)^3-7r^4m^8+9r^8m^4-r^{12}}{r(a^4+2r^4)^3}\right\}.$$

$$F=\text{force}=h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=\frac{h^2}{r^3}+\frac{h^2}{r^2}\cdot\frac{d^2u}{d\theta^2}.$$

$$\therefore F=\frac{h^2(7r^{12}+21r^8m^4+3r^4c^8-9c^4r^8-2c^4m^4r^4-m^8r^4)}{r^3(m^4-c^4+2r^4)^3}$$

$$=\frac{h^2\{7r^9+21m^4r^5-9c^4r^5+3rc^8-2c^4m^4r-m^8r\}}{(m^4-c^4+2r^4)^3}$$

41. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

If the earth were a perfect sphere and had a frictionless surface, what would be the motion of a ball placed at a given latitude?

[No solution of this problem has been received. EDITOR].

42. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

Find the time of vibration of a particle slightly displaced from the center of a solid cylinder in direction of the axis, the matter of the cylinder attracting according to the laws of nature.

Solution by the PROPOSER.

Consider the particle displaced by the amount x from the center of the cylinder. The matter attracting it will be a cylinder of length $2x$ at the opposite end of the cylinder. Call y the distance of any particle from the axis of cylinder and z the distance of particle from the end of cylinder with length $2x$ measured from the end towards the center.

The attraction of the cylinder upon the particle displaced from the center is

$$A=2\pi\int_0^R\int_0^{2x}\frac{y(a-x+z)dydz}{[(a-x+z)^2+y^2]^{\frac{3}{2}}}. \qquad A=2\pi\int_0^{2x}\left[dz-\frac{(a-x+z)dz}{[(a-x+z)^2+R^2]^{\frac{1}{2}}}\right].$$

$$=4\pi x-2\pi\sqrt{(a+x)^2+R^2}-2\pi\sqrt{(a-x)^2+R^2}$$

$$=4\pi x-2\pi\sqrt{(a^2+R^2)+(x^2+2ax)}-2\pi\sqrt{(a^2+R^2)+(x^2-2ax)}$$

$$=4\pi x-2\pi\{(a^2+R^2)^{\frac{1}{2}}+\tfrac{1}{2}\frac{x^2+2ax}{(a^2+R^2)^{\frac{1}{2}}}+\ldots\ldots\}$$

$$-2\pi\{(a^2+R^2)^{\frac{1}{2}}+\tfrac{1}{2}\frac{x^2-2ax}{(a^2+R^2)^{\frac{1}{2}}}\ldots\ldots\}$$

$$=4\pi x-4\pi(a^2+R^2)^{\frac{1}{2}}-2\pi\frac{x^2}{(a^2+R^2)^{\frac{1}{2}}}\ldots\ldots$$

Since the displacement is to be slight, we may neglect x^2 and all higher powers.

$\therefore (d^2x/dt^2)=4\pi x-4\pi(a^2+R^2)^{\frac{1}{2}}=cx-d$, for brevity.

$$\frac{d^2x}{dt^2}=\frac{dv}{dt}=\frac{dv}{dx}\cdot\frac{dx}{dt}=v\frac{dv}{dt}. \quad \therefore v\frac{dz}{dx}=cx-d. \quad v^2=cx^2-2dx+k.$$

When the displacement is a maximum the particle is at rest. Call the amplitude α.

$$\text{Then } t=\int_0^\alpha\frac{dx}{\sqrt{cx^2-2dx-c\alpha^2+2d\alpha}}=\int_0^\alpha\frac{dx}{\sqrt{(2d\alpha-c\alpha^2)+(cx^2-2dx)}}$$

$$=\int_0^\alpha[(2d\alpha-c\alpha^2)+(cx^2-2dx)]^{-\frac{1}{2}}dx=\frac{\alpha}{\sqrt{2d\alpha-c\alpha^2}},$$

neglecting higher powers of x.

$$=\sqrt{\frac{\alpha}{8\pi\sqrt{a^2+R^2}}}, \text{ neglecting the square of } \alpha.$$

V. Solution by A. H. HOLMES, Brunswick, Maine.

$x^2+(x+1)^2=\square$ or $2x^2+2x+1=\square$. Let $x=y+p$.

$\therefore$ $2y^2+(4p+2)y+2p^2+2p+1=\square$, from which we find the law of the series to be: $b=1+3a+2\sqrt{2a^2+2a+1}$. Let $a=3$ and we find $b=20$. Then by the same law, $c=119$, $d=696$, $e=4059$, and $f=23660$. Therefore, we have for the first six sets of values: (3 and 4), (20 and 21), (119, 120), (696 and 697), (4059 and 4060), and (23660 and 23661).

VI. Solution by H. C. WILKES, Skull Run, West Virginia.

We have $x^2+(x+1)^2=y^2=4n+1$, then $x(x+1)/2=n$. Substituting this value for n in $4n+1=y^2$ we have $x^2+x=(y^2-1)/2$. Putting $x+(x+1)=t$ or $x=(t-1)/2$, we obtain $t^2-2y^2=-1$. Since $t=7$, $y=5$ satisfy this equation, the first values of $x+(x+1)$ and y will be 7 and 5.

$\therefore$ $3^2+4^2=5^2$. From inspection of solution II, Problem 36, Vol. III, page 81, we find a formula for obtaining the succeeding values of $x+(x+1)$ and y.

$x++(x1)$.	y.	
$6\times7-1=41$,	$6\times5-1=29$,	$20^2+21^2=29^2$,
$6\times41-7=239$,	$6\times29-5=169$,	$119^2+120^2=169^2$,
$6\times239-41=1393$,	$6\times169-29=985$,	$696^2+697^2=985^2$,
$6\times1393-239=8119$,	$6\times985-169=5741$,	$4059^2+4060^2=5741^2$,
$6\times8119-2392=47321$.	$6\times5741-985=33461$.	$23660^2+23661^2=33461^2$.

Also solved by *J. SCHEFFER* and *G. B. M. ZERR*.

48. Proposed by B. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, O.

If any positive integral number N be divided by another positive integral number D, leaving a remainder 1, then any positive integral power of N, divided by D, will leave a remainder of 1.

I. Solution by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Let $N=nD+1$, then

$$(nD+1)^m=n^mD^m+mn^{m-1}D^{m-1}+\frac{m(m-1)}{2}n^{m-2}D^{m-2}+$$

$$\ldots\ldots+\frac{m(m-1)}{2}n^2D^2+mnD+1,$$

$$=D[n^mD^{m-1}+mn^{m-1}D^{m-2}+\frac{m(m-1)}{2}n^{m-2}D^{m-3}+\ldots\ldots+\frac{m(m-1)}{2}n^2D+mn]+1,$$

which proves the proposition.

Solved in a similar manner by *M. A. GRUBER* and *G. B. M. ZERR*.

II. **Solution by J. C. CORBIN, Pine Bluff, Arkansas ; P. S. BERG, Larimore, North Dakota ; E. W, MORRELL, Montpelier Seminary, Montpelier, Vermont ; A. P. READ, A. M., Clarence, Missouri ; and O. S. WESTCOTT, Principal North Chicago High School, Chicago.**

Put $N=nD+1$, then it is evident that if $N=nD+1$ be raised to any positive integral power, the last term will be 1 and every other term will contain D as a factor ; hence if this power be divided by D the remainder will be 1.

Also solved in a similar way by *A. H. BELL, JOSIAH H. DRUMMOND, ARTEMAS MARTIN* and *J. SCHEFFER.*

III. **Solution by J. O. MAHONEY, B. E., M. S., Graduate Fellow and Assistant in Mathematics, Vanderbilt University, Nashville, Tennessee.**

If $a\equiv a'$, $b\equiv b'$, $c\equiv c'$, $d\equiv d'$, etc., mod(D),

then $abcd\ldots\ldots\equiv a'b'c'd'\ldots\ldots$mod$(D)$.

Let $a=b=c=d$, etc.,$=N$, and $a'=b'=c'=d'$, etc.,$=1$, then $N^k\equiv 1$ mod(D).

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

39. **Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.**

A man is at the center of a circular desert; he travels at a given rate but in a *perfectly* random manner. What is the probability that he will be off the desert in a given time?

No solution of this problem has been received.

40. **Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.**

If every point of an ellipse be joined with every other point, what is the average length of the chords thus drawn ?

Solution by the PROPOSER.

Let $a\cos\theta$ and $(b/a)\sin\theta$ be the coördinates of one point, and $a\cos\phi$ and $(b/a)\sin\phi$ those of another.

The length of the chord joining them is

$$K=[a^2(\cos\phi-\cos\theta)^2+\frac{b^2}{a^2}(\sin\phi-\sin\theta)^2]^{\frac{1}{2}}\ .$$

Let s_1 and s_2=lengths of elliptic arcs from point $(a,\ 0)$ to points $(a\cos\theta,\ \frac{a}{b}\sin\theta)$ and $(a\cos\phi,\ \frac{a}{b}\sin\theta)$ respectively, and let S=whole distance around the ellipse.

Then $\frac{ds_1}{d\theta}=a(1-e^2\cos^2\theta)^{\frac{1}{2}}$ and $\frac{ds_2}{d\phi}=a(1-e^2\cos^2\phi)^{\frac{1}{2}}$.

Then the required average is

$$A=\frac{H}{S^2}\int_0^{\frac{1}{2}\pi}\int_0^{2\pi}Kds_1\,ds_2=\frac{4a^2}{S^2}\int_0^{\frac{1}{2}\pi}\int_0^{2\pi}[a^2(\cos\phi-\cos\theta)^2+\frac{b^2}{a^2}(\sin\phi-\sin\theta)^2]^{\frac{1}{2}}\times$$

$$(1-e^2\cos^2\theta)^{\frac{1}{2}}(1-e^2\cos^2\phi)^{\frac{1}{2}}\,d\theta d\phi.$$

This equation cannot be integrated in general terms.

Solved in the same manner by *G. B. M. ZERR*,

41. **Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.**

A line is drawn at random across the chord and given arc of a circular segment. Find the mean area of the divisions.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let A=area of given segment, A_1, A_2 mean areas of the two divisions.

$\therefore A_1+A_2=A$.

But, since the line is a random line, $A_1=A_2$.

$\therefore A_1=A_2=\frac{1}{2}A$.

Also solved by *HENRY HEATON*.

42. **Proposed by CHARLES E. MYERS, Canton, Ohio.**

A attends church 4 Sundays out of 5; B, 5 Sundays out of 6; and C, 6 Sundays out of 7. What is the probability of an event that A and B will be at church and C will not?

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas, and B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

The chance that A attends church$=\frac{4}{5}$.
The chance that B attends church$=\frac{5}{6}$.
The chance that C attends church$=\frac{6}{7}$.
The chance that A is not at church$=\frac{1}{5}$.
The chance that B is not at church$=\frac{1}{6}$.
The chance that C is not at church$=\frac{1}{7}$.
The chance that A and B attend and C not$=p_1=\frac{4}{5}\cdot\frac{5}{6}\cdot\frac{1}{7}=\frac{2}{21}$.
The chance that A and C attend and B not$=p_2=\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{1}{6}=\frac{4}{35}$.
The chance that B and C attend and A not$=p_3=\frac{5}{6}\cdot\frac{6}{7}\cdot\frac{1}{5}=\frac{1}{7}$.
The chance that A attends and B and C not$=p_4=\frac{4}{5}\cdot\frac{1}{6}\cdot\frac{1}{7}=\frac{4}{210}$.
The chance that B attends and A and C not$=p_5=\frac{1}{5}\cdot\frac{5}{6}\cdot\frac{1}{7}=\frac{1}{42}$.
The chance that C attends and A and B not$=p_6=\frac{1}{5}\cdot\frac{1}{6}\cdot\frac{6}{7}=\frac{1}{35}$.
The chance that A, B and C attend$=p_7=\frac{4}{5}\cdot\frac{5}{6}\cdot\frac{6}{7}=\frac{4}{7}$.
The chance that A, B and C do not attend$=p_8=\frac{1}{5}\cdot\frac{1}{6}\cdot\frac{1}{7}=\frac{1}{210}$.
p_1=probability required.
Also $p_1+p_2+p_3+p_4+p_5+p_6+p_7+p_8=1$.

Also solved by *HENRY HEATON*.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

40. Proposed by F. M. PRIEST, Mona House, St. Loufs, Missouri.

Suppose two cylindrical iron shafts, each 6 inches in diameter and respectively, 20 and 40 feet in height, are both standing perpendicular at the sea level. They start to fall in still air, how long will it require each one to fall to a horizontal position ?

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Neglecting the atmosphere and supposing the cylinder to revolve about a diameter in the base, we get, if l is the length of an equivalent pendulum, from works on Mechanics the formula for the time of vibration of a pendulum,

$$t=\sqrt{\frac{l}{g}}\int_\theta^a \frac{d\theta}{\sqrt{\sin^2\frac{1}{2}a-\sin^2\frac{1}{2}\theta}}.$$

In this problem a is $180^\circ=\pi$, $\theta=90^\circ=\frac{1}{2}\pi$.

$$\therefore t=\sqrt{\frac{l}{g}}\int_{\frac{1}{2}\pi}^{\pi}\frac{d\theta}{\cos\frac{1}{2}\theta}=\left[2\sqrt{\frac{l}{g}}\log.\left\{\frac{\tan(\frac{1}{4}\pi+\frac{1}{4}a)}{\tan(\frac{1}{4}\pi+\frac{1}{4}\theta)}\right\}\right]_{\theta=\frac{1}{2}\pi}^{a=\pi}.$$

$\therefore$ $t=\infty$, which proves that in a perfectly vertical position they will not fall unless moved slightly from this position. Let $a=\pi-\delta$ where δ is very small.

$$\therefore t=2\sqrt{\frac{l}{g}}\log.\left\{\frac{\cot\frac{1}{4}\delta}{\tan 3\pi/8}\right\}.$$

Let $\delta=1'$, l=length of cylinder, b=radius of base.
$\therefore$ $l=(3b^2+4l^2)/6l=13.3349$ feet for first cylinder.
$l=26.66745$ feet for second cylinder.
$\therefore t=2(.644328)(3.153498)=4.0638$ seconds for first cylinder.
$t=2(.911177)(3.153498)=5.7468$ seconds for second cylinder.

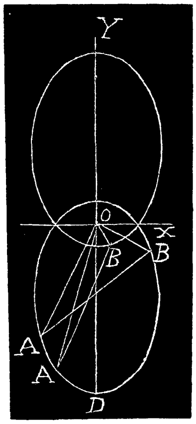
41. Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy, Pacific College, Santa Rosa, California.

A straight inflexible bar of uniform weight and thickness, length m is suspended at the two ends by a string without weight, length $l>m$ passing freely over a peg driven in a perpendicular wall. Describe and analyze the curve traced on the wall by the ends of the hanging bar.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let O be the peg, AB the rod. Let $OB=r$, $AO=r'$, $AB=m$, $\angle XOB=\theta$, $\angle XOA=\phi$.

Now in equilibrium OD always passes through the mid-point of AB.



Then $r+r'=1$ . . . .(1).

$m^2=r^2+r'^2-2rr'\cos(\phi-\theta)$ . . . .(2).

$r\cos\theta=r'\cos\phi$ . . . .(3).

(3) is obtained from the two triangles OAC, OBC.
(1) in (2) and (3) gives

$m^2=r^2+(1-r)^2-2r(l-r)\cos(\phi-\theta)$ . . . .(4).

$r\cos\theta=(1-r)\cos\phi$ . . . .(5).

(5) in (4) gives $l^2-m^2+2r^2\sin^2\theta-2rl=2r\sin\theta\sqrt{l^2-2rl+r^2\sin^2\theta}$.

$\therefore\ 4r^2(l^2-m^2\sin^2\theta)-4rl(l^2-m^2)+(l^2-m^2)^2=0.$

$$\therefore\ r=\frac{l^2-m^2}{2(l\pm m\sin\theta)}=\frac{l(1-e^2)}{2(1\pm e\sin\theta)},\text{ where } e=m/l.$$

This equation represents two equal ellipses with eccentricity $=m/l$, major axis $=l$, minor axis $=\sqrt{l^2-m^2}$, and O is one focus of each ellipse.

[The above solves the problem—"An ellipse confined to one vertical plane is suspended from a fixed point in space, coincident with a movable point on its circumference. Describe the curve marked out by foci." EDITOR].

EDITORIALS.

With January, 1897, the Chicago *Open Court* celebrates its decennial anniversary and now appears in the form of a monthly instead of a weekly.

Plane and Solid Analytical Geometry, by Frederick H. Bailey, A. M., and Frederick S. Woods, Ph. D., Assistant Professors of Mathematics in Massachusetts Institute of Technology, is announced as ready in March by Ginn and Company.

President H. H. Seerley, of the State Normal School of Cedar Falls, Iowa, has just ordered a complete set of the MONTHLY for the library. We only have a few more complete sets. Who wants them ?

We are in correspondence with several excellent mathematicians who are desirous of securing better positions for next year. If any of our readers know of such positions which are vacant or likely to become vacant at the end of this school year, we shall be pleased to refer them to these gentlemen.

With this number begins the fourth volume of the MONTHLY. No pains will be spared on the part of the Editors to make this volume better than any of the three previous ones, and in this effort they earnestly solicit the continued aid of all former contributors and subscribers. This number is sent to all our old subscribers, with bill enclosed, and anyone who may wish to discontinue should return this copy with his name written on the wrapper.

BOOKS AND PERIODICALS.

Elements of Analytical Geometry of Two Dimensions. The Fourteenth Edition. By Briot and Bouquet. Translated and Edited by James Harrington Boyd, Instructor in Mathematics in The University of Chicago. 8vo. Cloth, 582 pages. Introduction Price, $2. Chicago : Werner School Book Co.

This celebrated work so long known to mathematicians familiar with the French language, is now put in English dress, and is, therefore, at the service of American students. Comments on the material and the method of this work are unnecessary.

The work is divided into four books. Book I contains four chapters: Chapter I, Concerning Coördinates; Chapter II, Examples—The Circle, the Ellipse, the Hyperbola, the Parabola, Cissoid of Diocles, etc.; Chapter III, Concerning Homogenity; Chapter IV, Transformation of Coördinates. Book II contains three chapters: Chapter I, Straight Line; Chapter II, the Circle; Chapter III, the Geometrical Loci. Book III contains twelve chapters: Chapter I, Construction of Curves of the Second Degree; Chapter II, Center, Diameter, and Axes of Curves of the Second Degree; Chapter III, Reduction of the Equation of the Second Degree; Chapter IV, the Ellipse; Chapter V, the Hyperbola; Chapter VI, Concerning the Parabola; Chapter VII, Foci and Directrices; Chapter VIII, the Conic Sections; Chapter IX, the Determination of the Conic Sections; Chapter X, Theory of Poles and Polars; Chapter XI, General Properties of Conic Sections; Chapter XII, Secants Common to Two Conics. Book IV contains seven chapters: Chapter I, the Construction of Curves in Rectilinear Coördinates; Chapter II, Convexity and Concavity; Chapter III, Asymptotes; Chapter IV, Construction of Curves in Polar Coördinates; Chapter V, Concerning Similitude; Chapter VI, Graphic Solutions of Equations; Chapter VII, Notions Concerning Unicursal Curves.

From the table of contents it is seen that a leading feature of the work is its scope. It treats all the important methods invented by geometers, and includes some of the most beautiful discoveries of ancient and modern times. All subjects are treated in a practical way and illustrated by the applications of the theories to numerous problems. The book is beautiful as well as profound. The typographical and mechanical execution of the work is a credit to American text-book making. I very heartily commend this work to the careful consideration of teachers of Analytical Geometry and mathematical students desiring a good work on the subject. B. F. F.

The Outlines of Quaternions. By Lieutenant-Colonel H. W. L. Hime. 188 pages. Price, $3. Longmans, Green & Co. 1894. London and New York.

The first chapters deal with the properties of vectors. In the remaining pages we are introduced to quaternions proper,—their various forms and properties. The last chapter treats of the applications of quaternions to trigonometry, the triangle, the circle, conic sections, and other curves, the plane, tetrahedron, sphere and cone. These geometric applications show in some measure the usefulness of quaternions and give freshness and interest to the book. There is no preface. The addition of some exercises for solution would have added to the practical character of the work for class use. J. M. C.

Plane Surveying. By William G. Raymond, C. E., Member American Society of Civil Engineers; Professor of Geodesy, Road Engineering, and Topographical Drawing, in the Rensselaer Polytechnic Institute, Troy, New York. 8vo. Cloth, 486 pages (including tables). Price, $3. Chicago: American Book Co.

Some of the valuable features of this work are the detailed description of the use of instruments, accompanied by excellent illustrations and diagrams of the instruments themselves; the clear and comprehensible presentation of the subject matter of the work; and the fine form in which it appears for public favor. In its pages may be found treated plane table work and the use of the slide rule, planimeter and stadia measurements. Full tables and numerous examples of work in the way, both of underground and general topography are also given. B. F. F.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single numbers, 25 cents. The Review of Reviews Co., 13 Astor Place, New York City.

The *Review of Reviews* for February makes "A Plea for the Protection of Useful Men" from bores and "societies," and all well-meaning people who bother the life out of public men by letters and calls on the pretext of seeking assistance in some worthy undertaking. The editor of the *Review* publishes letters on this subject from the late Gen. Francis A. Walker, written only a few weeks before his death. In one of these letters General Walker wrote, "I am not well, and neither callars nor correspondents have any mercy.
B. F. F.

The Cosmopolitan. An Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1. per year in advance. Single number, 10 cents. The Cosmopolitan Co., Irvington-on-the-Hudson, New York.

The January number of the *Cosmopolitan* not only keeps up the usual literary excellence, artistic merit, and widest interests of that magazine, but also adds new features to its field of usefulness. The February number will contain the second part of Conan Doyle's new story.

During the year 1896, the *Cosmopolitan* reached the largest clientèle of intelligent, thoughtful readers possessed by any periodical in the world. The smallest issue of the year was 300,000 copies. B. F. F.

The Arena. A Monthly Magazine. Price, $3. Single number, 25 cents. Boston: Arena Publishing Co.

The *Arena* is the organ or mouth-piece of no one party, faction, or creed. It is unmortgaged and unbribed—a *free lance*, an *open arena*—wherein all honest and properly expressed and authoritative opinions, having in view the betterment of human conditions and human life, may be expressed. The best writers and authorities on leading questions contribute to its pages. Among the leading articles in the January number are the following: The Religion of Burns' Poems, by Rev. Andrew W. Cross; A Court of Medicine and Surgery, by A. B. Choate; Finance and Currency, by Gen. Heman Haupt; Daniel Webster's School Days, by Forest Prescott Hall; England's Hand in Turkish Massacres; etc.
B. F. F.

THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. IV. FEBRUARY, 1897. No. 2.

EMPIRICAL FORMULÆ FOR APPROXIMATE COMPUTATION.

By the Late ANSEL N. KELLOGG, of New York City.

[NOTE. The following paper is here printed in the exact form in which it was left by the author at the time of his death, except only a few necessary verbal alterations which are distinguished from the other parts of the paper by an enclosure of square brackets.]

The following symbols have the same signification, throughout these formulae, the only exception being when the letter "d" is used with a, b, c, etc.

$n=\dfrac{1}{m}=$index of root.

$m=\dfrac{1}{n}=$index of power.

$q=$quantity, or $\dfrac{q}{p}$, if fractional.

$s=$sum of $q+p$ (or the semi-sum).*

$d=$difference of q and p (or the demi-difference).*

$t=\dfrac{n^2-1}{3}$.

$U=$the sub-square, or under-square$=(\sqrt{q}-1)^2$.

$U_4=$the under-fourth$=(\sqrt[4]{q}-1)^4$, etc.

$E=$Napierian logarithm of number.

$K=$logarithm of number to base 4.

$r=$root of number.

*These last values for s and d cannot be used in the same equation with p and q.

=|= signifies nearly equal.

=||= signifies very nearly equal.

Mercator's formula for the extraction of roots of numbers near unity was equivalent to my own formula No. 1, as shown in Hutton's *Tracts on Mathematics*, Vol. I.

$$\frac{ns+d}{ns-d}=\sqrt[n]{q}, \text{ nearly} \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Hutton says he gave a formula for the correction of this result, but I have never been able to find it. Hutton himself gives the derivation of the above formula.

First correction of above:

$$\sqrt[n]{q} =|= \frac{ns+d-\frac{(n^2-1)d^2}{3ns}}{ns-d-\frac{(n^2-1)d^2}{3ns}}.$$

Substituting t for $\frac{n^2-1}{3}$, we have

$$\sqrt[n]{q} =|= \frac{ns+d-\frac{td^2}{ns}}{ns-d-\frac{td^2}{ns}} \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Second correction of above:

$$\sqrt[n]{\frac{q}{p}} =|= \frac{ns+d-\frac{t(d^2+[\sqrt{q}-\sqrt{p}]^4)}{ns}}{ns-d-\frac{t(d^2+[\sqrt{q}-\sqrt{p}]^4)}{ns}}; \ldots\ldots\ldots\ldots(3).$$

or by reducing,

$$\sqrt[n]{q} =|= \frac{(n^2+2)s+(2n^2-2)\sqrt{s^2-d^2}+3nd}{(n^2+2)s+(2n^2-2)\sqrt{s^2-d^2}-3nd} \ldots\ldots\ldots\ldots(4).$$

This nearly equals

$$\sqrt[n]{q} =|= \frac{ns+d-\frac{n^2-1}{3}\times\frac{d^2}{s^2-\frac{d^2}{4s}}}{ns-d-\frac{n^2-1}{3}\times\frac{d^2}{s^2-\frac{d^2}{4s}}} \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Simpler than these, and nearly related to (3) is

$$\sqrt[n]{q} =|= \frac{ns + d - \cfrac{td^2}{ns - \cfrac{td^2}{ns}}}{ns - d - \cfrac{td^2}{ns - \cfrac{td^2}{ns}}} \quad \ldots\ldots\ldots\ldots \quad (6).$$

Very simple and excellent is

$$\sqrt[n]{q} =|= \frac{ns + d - \cfrac{2tU}{n}}{ns - d - \cfrac{2tU}{n}} \quad \ldots\ldots\ldots\ldots \quad (7).$$

All the foregoing are what I call *equidistant* processes, because for all values of n, the difference between the numerator and denominator of the result is $2d$, (or some multiple) ; that is, the subtractive corrections following d are the same in both terms, whether q/p be a proper or an improper fraction.

Of the same nature, but differently derived, is formula (8) and its equivalent ($8\frac{1}{2}$).

$$\sqrt[n]{q} =|= \frac{n(2s^2 - d^2) + d^2 + 2ds}{n(2s^2 - d^2) + d^2 - 2ds} \ldots\ldots\ldots\ldots (8).$$

$$\sqrt[n]{q} =|= \frac{n(2s^2 - d^2) + d(2s + d)}{n(2s^2 - d^2) - d(2s - d)} \ldots\ldots\ldots\ldots (8\tfrac{1}{2}).$$

The fact that all these fractional formulae are symmetrical makes their operation comparatively simple.

In extracting roots of high numbers

$$\sqrt[n]{q} = 2^{\frac{m}{n}} \times \sqrt[n]{\frac{q}{2^m}},$$

and we are thus always enabled to use $q/2^m$ between the limits $\frac{1}{2}$ and 2. Hence, the following equation becomes of value :

$$\sqrt[n/m]{2} =|= \frac{3n + m - \cfrac{n^2 - m^2}{9n}}{3n - m - \cfrac{n^2 - m^2}{9n}}, \quad \ldots\ldots\ldots\ldots \quad (9),$$

$$\text{or } \sqrt[n/m]{2} =\!|\!= \frac{3n+m-\dfrac{n^2-m^2}{9n-\dfrac{n^2-m^2}{3n}}}{3n-m-\dfrac{n^2-m^2}{9n-\dfrac{n^2-m^2}{3n}}}, \quad \ldots\ldots\ldots\ldots\ldots (10),$$

$$\text{or } \sqrt[n/m]{2} =\!|\!= \frac{3n+m-\dfrac{29(n^2-m^2)}{507n}}{3n-m-\dfrac{29(n^2-m^2)}{507n}}, \quad \ldots\ldots\ldots\ldots\ldots (11).$$

In a latent form, the equidistant principle is also present in the following :

$$\sqrt[n]{\frac{q}{p}} =\!|\!= \frac{(2n^2+3n+1)q^2+(8n^2-2)qp+(2n^2-3n+1)p^2}{(2n^2-3n+1)q^2+(8n^2-2)qp+(2n^2+3n+1)p^2} \ldots\ldots (13).$$

In some of the following applications p is taken $=1$.
This [viz. (13)] becomes :

$$\text{for } \sqrt[2]{q} \ldots\ldots\ldots\ldots \frac{5q^2+10q+1}{q^2+10q+5} \ldots\ldots\ldots\ldots (13-2)$$

$$\text{for } \sqrt[3]{q} \ldots\ldots\ldots\ldots \frac{14q^2+35q+5}{5q^2+35q+14} \ldots\ldots\ldots\ldots (13-3)$$

$$\text{for } \sqrt[4]{\frac{q}{p}} \ldots\ldots\ldots\ldots \frac{15q^2+42qp+7p^2}{7q^2+42qp+15p^2} \ldots\ldots\ldots\ldots (13-4)$$

$$\text{for } \sqrt[5]{q} \ldots\ldots\ldots\ldots \frac{11q^2+33q+6}{6q^2+33q+11} \ldots\ldots\ldots\ldots (13-5)$$

$$\text{for } \sqrt[6]{q} \ldots\ldots\ldots\ldots \frac{91q^2+286q+55}{55q^2+286q+91} \ldots\ldots\ldots\ldots (13-6)$$

$$\text{for } \sqrt[7]{\frac{q}{p}} \ldots\ldots\ldots\ldots \frac{20q^2+65qp+13}{13q^2+65qp+20} \ldots\ldots\ldots\ldots (13-7)$$

$$\text{for } \sqrt[8]{q} \ldots\ldots\ldots\ldots \frac{51q^2+170q+35}{35q^2+170q+51} \ldots\ldots\ldots\ldots (13-8)$$

$$\text{for } \sqrt[9]{q} \ldots\ldots\ldots\ldots \frac{95q^2+323q+68}{68q^2+323q+95} \ldots\ldots\ldots\ldots (13-9)$$

for $\sqrt[10]{\dfrac{q}{p}}$ . . . . $\dfrac{77q^2+266qp+57p^2}{57q^2+266qp+77p^2}$ . . . . (13—10).

for $\sqrt[100]{q}$ . . . . $\dfrac{6767q^2+26666q+6567}{6567q^2+26666q+6767}$ . . . . (13—100).

For very high indices use, without sensible error,

$$\sqrt[n]{\frac{q}{p}} =|= \frac{(2n+3)q^2+8npq+(2n-3)p^2}{(2n-3)q^2+8npq+(2n+3)p^2} \quad\ldots\ldots\ldots\ldots\ldots (14).$$

(14) is the equivalent of (2).

Upon the logarithmic function depend the following formulæ :

$$\sqrt[n]{q} =|= \frac{n+\dfrac{E}{2}+\left(\dfrac{E}{6}\times\dfrac{r-1}{r+1}\right)}{n-\dfrac{E}{2}+\left(\dfrac{E}{6}\times\dfrac{r-1}{r+1}\right)} \quad\ldots\ldots\ldots\ldots\ldots (15).$$

$$\sqrt[n]{q} =|= \frac{3+\dfrac{2E}{n}+\dfrac{E^2}{2n^2}}{3-\dfrac{E}{n}} \quad\ldots\ldots\ldots\ldots\ldots (16).$$

E, the logarithm of $q =|= \dfrac{3(q^2-1)}{(q+1)^2+2q}$ . . . . (17),

or, if q be fractional, $=|= \dfrac{3(q^2-p^2)}{(q+p)^2+2qp}$ . . . . (18),

or, in terms of d and $s =|= \dfrac{6ds}{3s^2-d^2}$ . . . . (19).

This value of E, if q be between .9 and 1.1, is true to the seventh decimal, but may be corrected with very great accuracy, even up to the ninth or tenth decimal, by adding to the result the $\dfrac{4d^2}{45s^2}$th part of itself.

If, however q be so great as

1.7 or so small as .6 use 44 in place of 45
1.8 or so small as .56 use 43 in place of 45
1.9 or so small as .53 use 42 in place of 45
2. or so small as .5 use 41 in place of 45.

The accuracy of these formulæ will appear from the natural logarithms in the next [paragraph].

[*Some verifications of formula* (18) :

For $q/p=\frac{11}{10}$:—

By the formula : $\log(q/p)=\frac{63}{661}=.0953101$,

error=1 in last place.

By the tables: 2.3978951

2.3025851

$\log(q/p)=$ .0953100

For $q/p=\frac{16}{15}$:—

By the formula : $\log(q/p)=\frac{93}{1441}=.0645385$,

correct to last place.

By the tables : 2.7725887

2.7080502

.0645385

For $q/p=\frac{101}{100}$:—

By the formula : $\log(q/p)=\frac{603}{60601}=.0099\,5033$.]

Reconverting this by (15) we have

$$\frac{1+.004975165+.000008251}{1-.004975165+.000008251}=\frac{1.004983416}{.995033086}.$$

To this denominator add the 100th part,

$$\begin{array}{r} .995033086 \\ .009950331 \\ \hline 1.004983417 \end{array}$$

Hence the real number is $\frac{101}{100}$.

From the foregoing we have another value of $\sqrt[n]{q}$ as follows :

$$\sqrt[n]{q}=\frac{2n+\left(3+\frac{r-1}{r+1}\right)\left(\frac{q^2-p^2}{q^2+4qp+p^2}\right)}{2n-\left(3-\frac{r-1}{r+1}\right)\left(\frac{q^2-p^2}{q^2+4qp+p^2}\right)} \quad \ldots\ldots\ldots\ldots\ldots (20).$$

(Of course r can only be taken crudely, but may by successive steps, until the required approximation is reached.)

Another formula akin to (15), for values of q (or r) in terms of E is

$$\sqrt[n]{q}=\frac{n+\frac{E}{2}+\frac{E^2}{12}}{n-\frac{E}{2}+\frac{E^2}{12}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (15\tfrac{1}{2}).$$

Let us call c, which is equal to $\frac{q-1}{E}$, a *root centre*, meaning thereby that, for all values of q, and degrees of n, $\dfrac{c+\frac{q-1}{q+1}}{c-\frac{q-1}{q+1}}$ nearly equals $\sqrt[n]{q}$. Then

$$\frac{nc+\frac{q-1}{2}+\frac{(q-1)(r-1)}{6(r+1)}}{nc-\frac{q-1}{2}+\frac{(q-1)(r-1)}{6(r+1)}}=|=\sqrt[n]{q} \quad \ldots\ldots\ldots\ldots\ldots\ldots (21).$$

c, if q is under 10, is nearly $\dfrac{2\sqrt{q}+\frac{q+1}{2}}{3}$ $\ldots\ldots\ldots\ldots\ldots\ldots$ (22).

c, if q is between 50 and 250, is nearly same $-\dfrac{q^{\frac{3}{2}}}{500}$ $\ldots\ldots\ldots\ldots\ldots\ldots$ (23).

c, if q is between 10 and 50 [correction not given].

FACTOR PROCESS.

Take $\frac{b}{a}\times\frac{c}{b}\times\ldots\ldots\times\frac{h}{g}=q$; then $\frac{h}{a}=q$ and the series is consecutive.

Take $\frac{b}{a}\times\frac{d}{c}\times\ldots\ldots\times\frac{h}{g}=q$; then $\frac{bd\ldots h}{ae\ldots g}=q$.

Now if, to take the nth root of q, we assume n terms, consecutive or non-consecutive, and nearly equal in value

$$\sqrt[n]{q}=\frac{\frac{\sqrt{b}}{\sqrt{a}}+\frac{\sqrt{d}}{\sqrt{c}}+\ldots\ldots+\frac{\sqrt{h}}{\sqrt{g}}}{\frac{\sqrt{a}}{\sqrt{b}}+\frac{\sqrt{c}}{\sqrt{d}}+\ldots\ldots+\frac{\sqrt{g}}{\sqrt{h}}}, \quad \ldots\ldots\ldots\ldots (24),$$

wherein the numerators are the square roots of the terms and the denominators the reciprocals thereof.

If the terms are consecutive and odd in number

$$\sqrt[n]{q}=\frac{ah+2bg+\ldots\ldots+2de}{2ag+2bf+\ldots\ldots d^2}; \quad \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

but if consecutive and even in number

$$\sqrt[n]{q}=\frac{ai+2bh+\ldots\ldots\ldots+e^2}{2ah+2bg+\ldots\ldots\ldots+2de}\ \ldots\ldots\ldots\ldots\ldots\ (25\tfrac{1}{2}).$$

Also if the series is consecutive,

$$\sqrt[n]{q}=\frac{(g+h)b+(f+g)c+\ldots\ldots\ldots+(b+c)g+(a+b)h}{(g+h)a+(f+g)b+\ldots\ldots\ldots+(b+c)f+(a+b)h}.$$

(25) and ($25\frac{1}{2}$) are called the *diagonal* process.

To separate a quantity q/p into consecutive factors which shall be nearly equal, and which shall be as many as there are units in the index of the root to be extracted, and whose differences shall also be in arithmetical progression, use for expansor $\frac{n^2 s}{d^2}$, and then arrange terms with differences themselves differing by unity. This may be done by dividing d', the difference of the expanded fraction q'/p', or $q'-p'$, by n, and making the final interval $\frac{n-1}{2}$ less than the quotient, —after which ascend accordingly.

Illustrations: 1. Separate $\frac{2}{1}$ into three factors of above nature :—

$$\frac{n^2 s}{d^2}=13.5.\quad \text{Then}\ \frac{q'}{p'}=\frac{27}{13.5},\qquad \frac{13.5}{3}=4.5,\qquad \frac{n-1}{2}=1.$$

Then first interval=3.5. Therefore :

$$\frac{17}{13.5},\quad \frac{21.5}{17},\quad \frac{27}{21.5},\ \text{or}\ \frac{34}{27},\quad \frac{43}{34},\quad \frac{54}{43}$$

are the desired terms.

Applying the diagonal process :

$$\frac{1458+1462+1462}{1161+1161+1156}=\frac{4382}{3478}=\frac{2191}{1739}=1.2599195.\quad \text{Error}=.0000015.$$

2. Separate $\frac{5}{3}$ into 4 factors :—
Expansor=16 gives $\frac{5}{3}=\frac{80}{48}$; first interval=6.5.
[The series of consecutive factors is]

$$\frac{54.5}{48},\quad \frac{62}{54.5},\quad \frac{70.5}{62},\quad \frac{80}{70.5},\ \text{or}\ \frac{109}{96},\quad \frac{124}{109},\quad \frac{141}{124},\quad \frac{160}{141}.$$

3. Separate $\frac{3}{1}$ into five factors :—
Expansor=12.5 ; first interval=3, [and the series of consecutive factors is]

$$\frac{15.5}{12.5},\quad \frac{19.5}{15.5},\quad \frac{24.5}{19.5},\quad \frac{30.5}{24.5},\quad \frac{37.5}{30.5}, \text{ or } \frac{31}{25},\quad \frac{39}{31},\quad \frac{49}{39},\quad \frac{61}{49},\quad \frac{75}{61}.$$

If the factors are consecutive and in arithmetical progression, then

$$\sqrt[n]{q}=\frac{n^2q+\dfrac{n^2-1}{6}(q-1)^2}{n^2q-n(q-1)+\dfrac{(n-1)(n-2)}{6}(q-1)^2} \quad \ldots\ldots\ldots\ldots (27).$$

All quantities p/q may be represented in n terms, which group in three classes as follows:

$$\left(\frac{b}{a}\right)^j\times\left(\frac{d}{c}\right)^k\times\left(\frac{b}{e}\right)^l=q/p, \text{ where } j+k+l=n.$$

Then $$\sqrt[n]{q}=\frac{j(fc+de)b+k(fa+be)d+l(ad+bc)f}{j(fc+de)a+k(fa+be)c+l(ad+bc)e} \quad \ldots\ldots\ldots\ldots (28).$$

The above is called the *three-class* process.

Sometimes a quantity will reasonably resolve into n terms, which group in only two classes. Hence the following *two-class* process:

$$\left(\frac{b}{a}\right)^j\times\left(\frac{d}{c}\right)^k=q;$$

$$\sqrt[n]{q}=\frac{j(ad+bc+2cd)b+k(ad+bc+2ab)d}{j(ad+bc+2cd)a+k(ad+bc+2ab)c} \quad \ldots\ldots\ldots\ldots (29).$$

Or, the following, simpler but not so good:

$$\sqrt[n]{q}=\frac{(c+d)jb+(a+b)kd}{(c+d)ja+(a+b)kc} \quad \ldots\ldots\ldots\ldots (30).$$

Of all these processes the three-class (28) is the most trust-worthy. When q does not naturally resolve in such terms, take q/pv which does, and extract $\sqrt[n]{v}$, and multiply results.

In the consecutive series

$$\frac{b}{a}\times\frac{c}{b}\times\ldots\ldots\ldots\times\frac{e}{d}\times\frac{b}{e}=q,$$

take a new term, of the first expanded q times, that is qb/qa. Then drop the term b/a, and call $qb=g$ and $qa=f$, giving the equation

$$\frac{c}{b} \times \frac{d}{c} \times \ldots \ldots \ldots \times \frac{f}{e} \times \frac{g}{f} = q.$$

Now apply the diagonal process, according to the spirit, and not the letter, of (25), and we have,

$$\frac{bg+2cf+2de}{2bf+2ce+d^2} =|= \sqrt[n]{q} \quad \ldots\ldots\ldots\ldots (25\tfrac{1}{2}).$$

Now this result will be found no nearer than the result in (25) but the mean of the two will give a close result. It is not, however, a formula of value, and I think there are cases where the error of (25) and (25½) are both on the *same* side. Hence, their mean would be of no value in particular.

Process for Special Roots.

For Square Root: $\left[\text{Hutton gives } \frac{ac+b^2}{2ab} = \sqrt{\frac{b}{a} \times \frac{c}{b}}\right].$

$$\sqrt[2]{q} =|= \frac{(c+d)b+(a+b)d}{(c+d)a+(a+b)c} \quad \ldots\ldots\ldots\ldots (31).$$

$$\sqrt[2]{q} =|= \frac{3bcd+3abd+b^2c+d^2a}{3abc+3acd+a^2d+b^2c} \quad \ldots\ldots\ldots\ldots (32).$$

$$\sqrt[2]{q} =|= \frac{(\sqrt{cd} \times b)+(\sqrt{ab} \cdot d)}{(\sqrt{cd} \times a)+(\sqrt{ab} \times c)},$$

$$\text{or } =|= \frac{b\sqrt{cd}+d\sqrt{ab}}{a\sqrt{cd}+c\sqrt{ab}} \quad \ldots\ldots\ldots\ldots (33).$$

If we call r an approximate value of the required square root, then

$$\frac{q+r}{p+r} =|= \sqrt[2]{\frac{q}{p}} \quad \ldots\ldots\ldots\ldots (34).$$

$$\text{or } \frac{q^2+6r^2q+r^4}{4r(q+r^2)} =|= \sqrt[2]{q} \quad \ldots\ldots\ldots\ldots (35).$$

For Cube Roots:—

$$\frac{7q^3+42q^2+30q+2}{2q^3+30q^2+42q+7} =|= \sqrt[3]{q} \quad \ldots\ldots\ldots\ldots (36).$$

Also, if $\frac{b}{a} \times \frac{d}{c} \times \frac{f}{e} = q$,

$$\sqrt[3]{q} = | = \frac{bdc+bcf+adf}{acf+ade+bce} \text{ .... } (37).$$

This is the best of all cube root processes, and I call it the *interweaving* process. It has remarkable properties.

For the $\frac{3}{2}$ *Root* :—If $\frac{b}{a} \times \frac{c}{b} \times \frac{d}{c} = q$; then

$$\frac{2bd+c^2}{2ac+b^2} = | = \sqrt[\frac{3}{2}]{q}, \text{ or } \sqrt[3]{q^2} \text{ .... } (38).$$

For the Fourth Root:—Let

$$\frac{b}{a} \times \frac{c}{b} \times \frac{d}{c} \times \frac{e}{d} = q.$$

Then $\sqrt[4]{q} = | = \frac{(be+cd)b+(ae+bd)e+(ad+bc)d+(ac+b^2)e}{(be+cd)a+(ae+bd)b+(ad+bc)c+(ac+b^2)d}$ .... (39).

UNDER-SQUARE FORMULÆ.

$U = (\sqrt{q} - \sqrt{p})^2$, $U_4 = (\sqrt[4]{q} - \sqrt[4]{p})^4$, $U_8 = (\sqrt[8]{q} - \sqrt[8]{p})^8$ ....

(7) is an under-square formula and, if q is an even square, is precisely equivalent to the $\frac{1}{2}n$th root of $\sqrt{q}$, by the first correction of formula 1.

The first correction of (7) is

$$\sqrt[n]{\frac{q}{p}} = | = \frac{ns+d - \frac{2(n^2-1)}{3n} U - \frac{n^2-4}{6n} U_4}{ns-d - \frac{2(n^2-1)}{n} U - \frac{n^2-4}{6n} U_4} \text{ .... } (40).$$

Also, $$\sqrt[n]{\frac{q}{p}} = | = \frac{ns+d - \frac{2(n^2-1)}{3n} U - \frac{(E-1)U}{72} \times \frac{n(n-2)}{n-1}}{ns-d - \frac{2(n^2-1)}{3n} U - \frac{(E-1)U}{72} \times \frac{n(n-2)}{n-1}} \text{ ....} (41).$$

Still another but complex value of $\sqrt[n]{\frac{q}{p}}$ is :

$$\frac{ns+d - \frac{(2n^2-n)}{3n} U - \frac{(n^2-4)\ U_4}{6n} - \frac{(n^2-16)(\sqrt[8]{q} - \sqrt[8]{p})^4 \times (\sqrt{q}+1)\sqrt{p}}{{}^*12n}}{ns-d - \frac{(2n^2-n)}{3n} U - \frac{(n^2-4)\ U_4}{6n} - \frac{(n^2-16)(\sqrt[8]{q} - \sqrt[8]{p})^4 \times (\sqrt{q}+1)\sqrt{p}}{{}^*12n}} \text{ .. } (42).$$

*It seems that $11n$ is better in actual practice.

For cube root use for coefficients $\frac{16}{9}$ and $\frac{5}{19}$ instead of $\frac{16}{9}$ and $\frac{5}{18}$.

Illustrations: All the equidistant formulæ, except these, approach accuracy only when q is near unity. No such restriction binds these [the under-square] formulæ, especially those which are most developed. And here, let me say, is undoubtedly the beginning of a series which I think the Calculi would unfold, and I trust some friend of science will take the burden of solving it. So developed, I am satisfied that all roots of all numbers would be extractable to any required degree of accuracy.

1. Taking (40) extract $\sqrt[3]{4096}$.

$$\text{Root}=\frac{3\times4097+4095-\frac{16}{9}\times3969-\frac{5}{18}\times2401}{3\times4097-4095-\frac{16}{9}\times3969-\frac{5}{18}\times2401}$$

$$=\frac{12291+4095-7056-667\text{ nearly}}{12291-4095-7056-667\text{ nearly}}=\frac{8663}{473}=18+;$$

but taking $\frac{5}{19}$ instead of $\frac{5}{18}=\frac{8696}{508}=17+$.

2. Taking $\sqrt[4]{4096}$, we have

$$\text{Root}=\frac{4\times4097+4095-\frac{5}{2}\times3969-\frac{1}{2}\times2401}{4\times4097-4095-\frac{5}{2}\times3969-\frac{1}{2}\times2401}=\frac{9360}{1170}=8.$$

3. Take $\sqrt[6]{4096}$. By (42),

$$\text{Root}=\frac{6\times4097+4095-\frac{35}{9}\times3969-\frac{8}{9}\times2401-\frac{5}{18}\times\frac{100}{9}\times65}{6\times4097-4095-\frac{35}{9}+3969-\frac{8}{9}\times2401-\frac{5}{18}\times\frac{100}{9}\times65}=\frac{11907.2}{3717}=3.2,$$

but should $=4$.

To Sum a Series: $S=1,\ r,\ r^2,\ \ldots\ldots\ r^{n-1}$, n terms.

Let $s=\text{ratio}+1=r+1$, $d=r-1$.

$$S=|=\frac{2n}{s-nd+\dfrac{(n^2-1)d^2}{3s}}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ (43).$$

If s exceeds 2, this is not accurate enough to be of value.

Cube Root by Difference Method.

Take $a^3<q$, $b^3>q$. Call $q-a^3=A$, $b^3-q=B$. Then

$$\frac{a^2B+b^2A}{aB+bA}=|=\sqrt[3]{q}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (44).$$

Convenient Formula for Roots of 2.

$$\sqrt[n]{2} \mathrel{=\!|\!=} \frac{101n+35+\dfrac{35(r-1)}{6(r+1)}}{101n-35+\dfrac{35(r-1)}{6(r+1)}} \dots\dots\dots\dots (45).$$

A Rough Value of q in Terms of E.

$$q \mathrel{=\!|\!=} \frac{\sqrt{9+3E^2}+2E}{3-E} \dots\dots\dots\dots (46).$$

Logarithms.

A curious fact, but scarcely a useful one, is: The logarithm of any number approaches

$$K_B \mathrel{=\!|\!=} \frac{\sqrt{B}\times(q-B^m)}{\sqrt{B}\times(\sqrt{B}-1)B^m+(\sqrt{B}-1)q}+m, \dots\dots\dots\dots (47),$$

wherein K is the logarithm, B the base of the system, m the characteristic of the logarithm of q, the quantity. Now if $B=4$, the formula becomes

$$K_4 \mathrel{=\!|\!=} \frac{2(q-B^m)}{2B^m+q}+m, \dots\dots\dots\dots (48).$$

If $q=q/p$, then $K_4 \mathrel{=\!|\!=} \dfrac{2q-2B^mp}{2B^mp+q}+m \dots\dots\dots\dots (49).$

If $K_4=\dfrac{m+c}{d}$, then $q \mathrel{=\!|\!=} B^m(1+\dfrac{3c}{2d-c}) \dots\dots\dots\dots (50)$

$$\mathrel{=\!|\!=} B^m\frac{2d+2c}{2d-c} \dots\dots\dots\dots (51).$$

Since Napierian $\log E \mathrel{=\!|\!=} \frac{44}{61}K$, and also, since

$$\sqrt[n]{q} \mathrel{=\!|\!=} \frac{\dfrac{44}{61}K+\dfrac{1}{2n}}{\dfrac{44}{61}K-\dfrac{1}{2n}}$$

roughly, then

$$\sqrt[n]{q} =|= \frac{\frac{44}{61}K+\frac{1}{2n}}{\frac{44}{61}K-\frac{1}{2n}} \text{ roughly} \ldots\ldots\ldots\ldots\ldots\ldots(52).$$

The term K also may be understood to mean the logarithm of q to base 4.

K of $10=\frac{5}{3}$; then $q=4(1+\frac{6}{4})=10$.

The error by this method is easily represented by a curve.

In the fractional processes continually occur the coaddition of fractions, or the adding of numerators together, and of denominators also, when the latter are not the same. When the fractions, two or more, are "embracing," the results are close, and if "harmoniously embracing" then positively accurate geometric averages. They are harmonious when the product of the two terms of each fraction are the same.

Thus, $\frac{8}{5}$ and $\frac{7}{6}$ are embracing, and their co-sum doubled is $2\times\frac{15}{11}=2.727+$. The real sum is $\frac{83}{30}=2.767+$. $\frac{12}{7}$, $\frac{11}{8}$ and $\frac{10}{9}$ are embracing, and their co-sum tripled is $3\times\frac{33}{24}=4.125$. The real sum is $\frac{2117}{504}=4.200$. $\frac{9}{4}$ and $\frac{6}{6}$ are harmoniously embracing. Their co-sum is $\frac{15}{10}$ or $\frac{3}{2}$ which is their square root and precise geometric average.

Rough addition may be performed by co-addition. Thus, $\frac{10}{17}+\frac{21}{33}=|=\frac{21}{33}$, or $\frac{7}{11}$, which doubled differs from $\frac{347}{272}$ as 3817 differs from 3808, or 1 in 423 parts.

In the foregoing processes for extraction of roots, calculation of logarithms, etc., excepting the factor-process, the accuracy increases rapidly as q approaches unity. In general they have value only when q lies between .5 and 2. Their value should be tested by logarithms. When q exceeds 2, it may generally be divided by the nth power of some simple quantity, which will bring it near unity. If this be difficult, use the nearest power of 2 (say m) as previously explained.

Many apparently absurd problems are readily answered by these formulæ. Thus: *If the cube root of* $2.5=\frac{19}{14}$ *what is the* 4*th root of* $3.333+$?

Using the 2-class process, formula (29),—

$$\left(\frac{19}{14}\right)^3\times\left(\frac{4}{3}\right)=\text{(say to } 3\tfrac{1}{3}\text{)}\ \frac{(125\times3\times19)+(645\times4)}{(375\times14)+(645\times3)}$$

$$=\frac{9705}{7160}=\frac{1941}{1432}=1.35545, \text{ [error here]}$$

or by (30),

$$\frac{399+132}{294+99}=\frac{531}{393}=\frac{177}{131}=1.35114.$$

True answer $=1.35120$.

Comparison of Fraction.

It is frequently necessary, or desirable to compare the values of two unwieldly, yet not very unequal fractions, or to ascertain approximately the comparative value of two ordinary fractions quickly, even if only approximately. Hence the following :

Compare b/a with d/c. Suppose them nearly equal to z/y, and divide each fraction by it, giving by/az and dy/cz. Subtract unity from each, and we have $\frac{by\pm az}{az}$ which make equal to $\frac{1}{uaz}$, and $\frac{dy-cz}{cz}$ which make equal to $\frac{1}{vcz}$. Then

$$\frac{uaz\mp vcz}{(uaz)\times(vcz)}=\frac{(ua\mp vc)z}{uavcz}$$

=the relative excess of b/a over d/c. If u and v are each unity, the process is very simple.

Illustrations : 1. Compare $\frac{5}{8}$ and $\frac{7}{11}$ with $\frac{2}{3}$ as measure.

$$(\tfrac{5}{8}\div\tfrac{2}{3})-1=-\tfrac{1}{16} \text{ and } (\tfrac{7}{11}\div\tfrac{2}{3})-1=-\tfrac{1}{22}.$$

$$\frac{22-16}{22\times16}=\frac{1}{58\frac{2}{3}}, \text{ or } =\frac{11-8}{11\times8\times2}=\frac{1}{58\frac{2}{3}}.$$

True answer$=\frac{1}{56}$.

2. Compare $\frac{39}{29}$ with $\frac{65}{49}$. Take base$=\frac{4}{3}$.

$$\frac{by}{az}=\frac{116}{117}, \quad \frac{dy}{cz}=\frac{196}{195}, \quad \frac{195+117}{195\times117}=\frac{312}{22815}=\frac{1}{73.1+}.$$

By the formula $\frac{49+29}{49\times29\times4}=\frac{76}{5684}=\frac{1}{74.8}$.

True answer$=\frac{1911-1885}{1911}=\frac{1}{73.5}$.

New York, 1878—1883.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

71. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

A man owes me \$200 due in 2 years, and I owe him \$100 due in 4 years; when can he pay me \$100 to settle the account equitably, money being worth 6%?

I. Solution by P. S. BERG, Principal of Schools, Larimore, North Dakota; and the PROPOSER.

Let x=the time.

Now, the present worth of \$200 for $(2-x)$ years—the present worth of \$100 for $(4-x)$ years must=\$100.

$\dfrac{10000}{56-3x}$=present worth of \$200 for $(2-x)$ years at 6%.

$\dfrac{5000}{62-3x}$=present worth of \$100 for $(4-x)$ years at 6%.

$$\therefore \frac{10000}{56-3x}-\frac{5000}{62-3x}=100.$$

$\therefore$ x=.358615 years=4 months and 9 days.

II. Solution by FREDERIC R. HONEY, New Haven, Connecticut.

The present value of \$1.12 due 2 years hence is \$1.00. Therefore the present value of \$200.00 due in 2 years is \$200÷1.12=\$178.571. The present value of \$1.24 due 4 years hence is \$1.00. Therefore the present value of \$100.00 due 4 years hence is \$100÷1.24=\$80.645. If we deduct \$80.645 from \$178.571 we have \$97.926, the amount due to me at the present time. This sum placed at interest at 6% would yield \$97.926×.06=\$5.876 in 1 year. The difference between \$100.00 and \$97.926 is \$2.074, the interest which must accumulate in order that the sum may become equal to \$1.00. Therefore since the interest \$5.876 accumulates in 1 year, the interest \$2.074 will accumulate in 2.074÷5.876=0.3529 years. Answer.

72. Proposed by W. H. CARTER, Professor of Mathematics, Centenary College of Louisiana, Jackson, Louisiana.

Though the length of my field is 1-7 longer than my neighbor's, and its quality is 1-9 better, yet as its breadth is 1-4 less, his is worth \$500 more than mine. What is mine worth? *Encyclopedia Britannica.*

Solution by Misses EVA JONES and NEVA CAROTHERS, Senior Pupils of West Point Graded School.

1. $l : l' :: 7 : 8$. 1st condition.

2. $q : q' :: 9 : 10$. 2nd condition.

3. $b : b' :: 4 : 3$. 3rd condition.

4. $v : v' :: 21 : 21$, multiplying and reducing, and remembering that the value $\propto l.b.q$.

5. Also $v-v'=\$500$. Whence,

6. $v=\$10500$. From (4) and (5),

7. $v'=\$10000$.

This problem was also solved by *B. F. SINE*, *NELSON S. RORAY*, *P. S. BERG*, *F. M. McGAW*, *J. C. CORBIN*, *COOPER D. SCHMITT*, *FREDERIC R. HONEY*, *H. C. WILKES*, and *G. B. M. ZERR*.

M. A. Gruber sent in a solution of Problem 70, Department of Arithmetic, too late for credit in last issue. His answer is 6.48 years.

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

68. Proposed by ROBERT JUDSON ALEY, M. A., Professor of Mathematics in Indiana University, Bloomington, Indiana.

Sum to n terms the series, $n\cos\theta+(n-1)\cos2\theta+(n-2)\cos3\theta$, etc.

[*Chrystal's Algebra.*]

I. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics in Columbian University, Washington, D. C.

Let $S=n\cos\theta+(n-1)\cos2\theta+(n-2)\cos3\theta$........ Also let $S_s=\sin\theta+\sin2\theta+\sin3\theta$........, and $S_c=\cos\theta+\cos2\theta+\cos3\theta$........

$$S=n[\cos\theta+\cos2\theta+\cos3\theta+........]-[\cos2\theta+2\cos3\theta+........],$$

$$=(n+1)[\cos\theta+\cos2\theta+\cos3\theta+........]-[\cos\theta+2\cos2\theta+3\cos3\theta........],$$

$$=(n+1)S_c-dS_s/d\theta.$$

Now $S_c=[\cos\{\frac{1}{2}(n+1).\theta\}\sin\frac{1}{2}(n\theta)]/\sin\frac{1}{2}\theta$, and $S_s=[\sin\{\frac{1}{2}(n+1).\theta\}\sin\frac{1}{2}(n\theta)]/\sin\frac{1}{2}\theta$.

$$\therefore S=(n+1)\frac{\cos\{\frac{1}{2}(n+1)\theta\}\sin\frac{1}{2}(n\theta)}{\sin\frac{1}{2}\theta}-\frac{d}{d\theta}\left[\frac{\sin\frac{1}{2}(n+1).\theta\sin\frac{1}{2}(n\theta)}{\sin\frac{1}{2}\theta}\right],$$

probably as compact a form as can be obtained.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Let $S=$sum required,

$$2\sin\tfrac{1}{2}\theta\cos n\theta=\sin\{\theta+\frac{2n-1}{2}\theta\}-\sin\{\theta+\frac{2n-3}{2}\theta\}$$

$$4\sin\tfrac{1}{2}\theta\cos(n-1)\theta=2\sin\{\theta+\frac{2n-3}{2}\theta\}-2\sin\{\theta+\frac{2n-5}{2}\theta\}$$

$$6\sin\tfrac{1}{2}\theta\cos(n-2)\theta=3\sin\{\theta+\frac{2n-5}{2}\theta\}-3\sin\{\theta+\frac{2n-7}{2}\theta\}$$

$$8\sin\tfrac{1}{2}\theta\cos(n-3)\theta=4\sin\{\theta+\frac{2n-7}{2}\theta\}-4\sin\{\theta+\frac{2n-9}{2}\theta\}$$

....

$$2n\sin\tfrac{1}{2}\theta\cos\theta=n\sin(\theta+\tfrac{1}{2}\theta)-n\sin(\theta-\tfrac{1}{2}\theta).$$

Adding we get

$$2S\sin\tfrac{1}{2}\theta=(\sin\tfrac{3}{2}\theta+\sin\tfrac{5}{2}\theta+\sin\tfrac{7}{2}\theta+\ldots\ldots\ldots+\sin\frac{2n+1}{2}\theta)-n\sin\tfrac{1}{2}\theta,$$

$$=[\sin(\frac{n+2}{2})\sin\tfrac{1}{2}(n\theta)]/\sin\tfrac{1}{2}\theta-n\sin\tfrac{1}{2}\theta.$$

$$\therefore\ S=[\sin(\frac{n+2}{2}\theta)\sin\tfrac{1}{2}(n\theta)-n\sin^2\tfrac{1}{2}\theta]/2\sin^2\tfrac{1}{2}\theta.$$

The series in parenthesis above is summed in all trigonometries in the series, $\sin\alpha+\sin(\alpha+\beta)+$, etc., by making $\alpha=\frac{3}{2}\theta$, $\beta=\theta$.

III. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

The given series may be broken up into:

$$n[\cos\theta+\cos2\theta+\cos3\theta+\ldots\ldots\ldots\cos n\theta]$$
$$-[\cos2\theta+2\cos3\theta+3\cos4\theta+\ldots\ldots\ldots(n-1)\cos n\theta].$$

To sum the series $\cos\theta+\cos2\theta+\cos3\theta+\ldots\ldots\ldots\cos n\theta$, we have

$\sin\frac{1}{2}\theta-\sin\frac{3}{2}\theta=-2\cos\theta\sin\frac{1}{2}\theta.$

$\sin\frac{3}{2}\theta-\sin\frac{5}{2}\theta=-2\cos2\theta\sin\frac{1}{2}\theta.$

$\sin\frac{5}{2}\theta-\sin\frac{7}{2}\theta=-2\cos3\theta\sin\frac{1}{2}\theta.$

....

$\sin\frac{1}{2}(2n-1)\theta-\sin\frac{1}{2}(2n+1)\theta=-2\cos n\theta\sin\frac{1}{2}\theta.$

Adding, we have, $\sin\frac{1}{2}\theta-\sin\frac{1}{2}(2n+1)\theta=-2\sin\frac{1}{2}\theta\Sigma(n\theta).$

$\therefore\ \Sigma(n\theta)=[\sin\frac{1}{2}(2n+1)\theta-\sin\frac{1}{2}\theta]/2\sin\frac{1}{2}\theta=[\cos\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta]/\sin\frac{1}{2}\theta.$

$\therefore\ n\Sigma(n\theta)=[n\cos\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta]/\sin\frac{1}{2}\theta.$

To sum the second part, we have,

$x^2+2x^3+3x^4+\ldots\ldots\ldots(n-1)x^n=[x^2-nx^{n+1}+(n-1)x^{n+2}]/(1-x)^2.$

Putting $x=\cos\theta+i\sin\theta$, and employing the formula $(\cos\theta+i\sin\theta)^m=\cos m\theta+i\sin m\theta$, we obtain after putting the real parts of both members equal, and making all necessary reductions, for the sum of the second series

$$=\frac{\cos\theta-n\cos n\theta+(n-1)\cos(n+1)\theta}{4\sin^2\frac{1}{2}\theta};$$

so that the sum of the given series

$$=\frac{n\cos\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta}+\frac{\cos\theta-n\cos n\theta+(n-1)\cos(n+1)\theta}{4\sin^2\frac{1}{2}\theta}.$$

To test this formula we must of course, leave the coefficient n of the first expression unchanged, while in all the other factors and terms which involve n, n must be put successively=1, 2, 3, 4, etc.

Also solved by *E. W. MORRELL*.

69. Proposed by C. E. WHITE, A. M., Trafalgar, Indiana.

Prove that $x^n\pm x^{n-1}+x^{n-2}\pm\ldots\ldots\ldots+(\pm1)^{n-1}x+(\pm1)^n=(x\mp1)^n\pm A(x\mp1)^{n-1}+B(x\mp1)^{n-2}\pm\ldots\ldots\ldots+(\pm1)^n x$, where A, B, C, ... are the binomial coefficients of the $(n+1)$th order.

I. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics in Columbian University, Washington, D. C.

$$x^n\pm x^{n-1}+x^{n-2}+\ldots\ldots\ldots,\text{ etc.}=(x^{n+1}-1)/(x-1)\ \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

or $x^{n+1}+1)/(x+1)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$, $=\{[(x-1)+1]^{n+1}-1\}/(x-1)$, or

$\{[(x+1)-1]^{n+1}+1\}/(x+1)$, $=(x-1)^n+C^2_{n+1}(x-1)^{n-1}+C^3_{n+1}(x-1)^{n-2}+\ldots\ldots$,

$$\text{or } (x+1)^n-C^2_{n+1}(x+1)^{n-1}+C^3_{n+1}(x-1)^{n-2}-\ldots\ldots,$$

$$=(x\mp1)^n\pm A(x\mp1)^{n-1}+B(x\mp1)^{n-2}+\ldots\ldots$$

II. Solution by E. W. MORRELL, Professor of Mathematics in Montpelier Seminary, Montpelier, Vermont.

Let $K=x^n\pm x^{n-1}+x^{n-2}\pm\ldots\ldots\ldots+(\pm1)^{n-1}x+(\pm1)^n$.

Put $x=y\pm1$, expanding and observing that the sign of the last term of each expression is $\pm$ if n is odd but $+$ if n is even, we may write:

$$x^n=(y\pm1)^n=y^n\pm ny^{n-1}+\tfrac{1}{2}[n(n-2)]y^{n-2}\pm\ \ldots\ldots\ldots\ldots\ldots+(\pm1)^{n-1}ny+(\pm1)^n$$

$$\pm x^{n-1}=\pm(y\pm1)^{n-1}=\pm y^{n-1}+(n-1)y^{n-2}\pm\ \ldots\ldots\ldots+(\pm1)^{n-1}(n-1)y+(\pm1)^n$$

$$x^{n-2}=(y\pm1)^{n-2}=\ldots\ldots\ldots\ldots\ldots\ldots\ldots y^{n-2}\pm\ldots\ldots\ldots\ldots+(\pm1)^{n-1}(n-2)y+(\pm1)^n$$

etc ....etc.

$(\pm1)^{n-1}x=(\pm1)^{n-1}(y\pm1)=$ .... $(\pm1)^{n-1}y+(\pm1)^n$

$(\pm1)^n=$...... .... $(\pm1)^n$.

By adding, and simplifying the coefficients of y, we have

$K=y^n\pm(n+1)y^{n-1}+\frac{1}{2}[(n+1)n]y^{n-2}\pm\ \ldots\ldots+(\pm1)^{n-1}\frac{1}{2}[(n+1)n]y+(\pm1)^n(n+1)$,

which has binomial coefficients of the $(n+1)$th order. Substituting A, B, C, ... for the coefficients and restoring the values of y,

$K=(x\mp1)^n\pm A(x\mp1)^{n-1}+B(x\mp1)^{n-2}\pm$........... $+(\pm1)^{n-1}B(x\mp1)+(\pm1)^nA$.

[Expanding and combining the terms of the second member, we get the first member for a result. ZERR.]

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

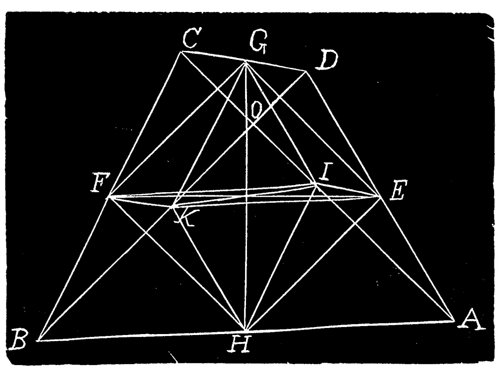
SOLUTIONS OF PROBLEMS.

43. Proposed by J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

The consecutive sides of a quadrilateral are a, b, c, d. Supposing its diagonals to be equal, find them and also the area of the quadrilateral.

II. Solution by A. H. BELL, Hillsboro, Illinois.

The solution as published simply demonstrates this theorem, $a^2+b^2+c^2+d^2=2x^2+4\overline{IK}^2$, with two unknowns, and is then solved for a particular case.



Let the sides of the quadrilateral AB, BC, CD, and AD, be a, b, c, and d; and the diagonals each $=2x$; $x+y$, $x-y$=the segments AO and OC; and BO and $OD=x+z$ and $x-z$. In the triangles AOB, BOC, and COD we have $(x+y)\cos A+(x+z)\cos B=a$, $\cos A=(4x^2+a^2-b^2)/(4ax)$, $\cos B=(4x^2+a^2-d^2)/(4ax)$, making

$z(4x^2+a^2-d^2)=(2a^2+b^2+d^2)x-8x^3-(4x^2+a^2-b^2)y$ ....(1).

Similarly BOC, $z(4x^2+b^2-c^2)=(2b^2+a^2+c^2)x-8x^3-(a^2-b^2-4x^2)y$ .....(2),

and COD, $z(-4x^2+b^2-c^2)=(b^2+2c^2+d^2)x-8x^3+(c^2-d^2+4x^2)y$ ....(3).

Subtracting (2) from (1), $(a^2-b^2+c^2-d^2)z=(a^2-b^2-c^2+d^2)x-8x^2y$ .....(4).

Subtracting (3) from (2), $8x^2z=(a^2+b^2-c^2-d^2)x-(a^2-b^2+c^2-d^2)y$.....(5).

Equating the values of z in (4) and (5) and solving for y, after letting

$(a^2-b^2+c^2-d^2)=e$, $a^2-b^2-c^2+d^2=f$, $a^2+b^2-c^2-d^2=g$, we have

$y=(eg-8fx^2)/(e^2-64x^4).x$......................(6).

Equating the values of z and solving for y in (1) and (4) after letting

$a^2-b^2=m,\quad a^2-d^2=n,\quad 2a^2+b^2+d^2=p,\quad 2e+f=q,$

and noting that $2n-e=g$, we have

$y=(4qx^2-ep+fn)/(32x^4+4gx^2-em).x$...............(7).

Equating the values of y in (6) and (7),

$$512x^6-64(a^2+b^2+c^2+d^2)x^4+16[a^2(b^2-2c^2+d^2)+b^2(c^2-2d^2)+c^2d^2]x^2$$
$$+4(ac-bd)(ac+bd)(a^2-b^2+c^2-d^2)=0 \text{.......(8).}$$

Let $x^2=\frac{1}{8}y$, $4x^2=\frac{1}{2}y$, $2x=\sqrt{\frac{1}{2}y}$, or multiply (8) by the geometrical series, $\frac{1}{512}$, $\frac{1}{64}$, $\frac{1}{8}$, and 1, ratio=8, then (8) becomes

$$y^3-(a^2+b^2+c^2+d^2)y^2+2[a^2(b^2-2c^2+d^2)+b^2(c^2-2d^2)+c^2d^2]y$$
$$+4(ac-bd)(ac+bd)(a^2-b^2+c^2-d^2)=0 \text{........(9).}$$

When this is solved by Cardan's formula, then since there are given the sides of the triangles ABC and ACD, we have in the general formula, with the sides a, b, and $2x$, and c, d, and $2x$, the area for the quadrilateral

$$ABCD=\tfrac{1}{4}\sqrt{[(a+b)^2-4x^2]\times[4x^2-(a-b)^2]}$$
$$+\tfrac{1}{4}\sqrt{[(c+d)^2-4x^2]\times[4x^2-(c-d)^2]}.$$

A better transformation for (8) is to put $4x^2=y$, $2x=\sqrt{y}$. We then get $y^3-\frac{1}{2}(a^2+b^2+c^2+d^2)y^2+\frac{1}{2}[a^2(b^2-2c^2+d^2)+b^2(c^2-2d^2)+c^2d^2]y+\frac{1}{2}(ac+bd)(ac-bd)(a^2-b^2+c^2-d^2)=0$.

Example: $a, b, c, d=6, 5, 3, 4$, respectively, in (9) gives

$$y^3-86y^2+894y-1216=0.$$

By Horner's method, $y=75.7270176+$. Diagonal $2x=6.1533+$ agreeing with a close drawing.

[Mr. Bell sent us this solution March 14, 1895. We have looked it over carefully and believe that it is entirely correct. The solution published in the July-August number of Vol. II is of a particular case. EDITOR.]

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

57. Proposed by F. M. McGAW, A. M., Professor of Mathematics in Bordentown Military Institute, Bordentown, New Jersey.

Solve the following equation: $(1+x^2)\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}+2y=0.$

I. Solution by WILLIAM E. HEAL, A. M., Member of the London Mathematical Society, and Treasurer of Grant County, Marion, Indiana.

Let $y=bx\left[\int zdx+c\right]$ and the equation becomes

$$x(1+x^2)\frac{dz}{dx}+2z=0, \text{ or } \frac{dz}{z}+\frac{2dx}{x(1+x^2)}=0.$$

$$\therefore z=c'(1+[1/x^2])\ ;\ y=bx\{c'(x-[1/x])+c\},\ =Bx+A(1-x^2).$$

II. Solution by H. C. WHITAKER, M. E., Ph. D., Professor of Mathematics in Manual Training School, Philadelphia, Pennsylvania.

Proceeding to obtain a solution in series, both values of y are found to terminate immediately. The complete primitive is $y=Ax+B(x^2-1)$.

III. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio State University, Athens, Ohio.

It is shown (*Forsyth's Differential Equations*, Article 58) that

$$d^2y/dx^2+P(dy/dx)+Qy=R\ \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

gives, when $y=vw$ ....(2).

$$w\frac{d^2v}{dx^2}+(2\frac{dw}{dx}+Pw)\frac{dv}{dx}+(\frac{d^2w}{dx^2}+P\frac{dw}{dx}+Qw)v=R \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

with the conditional equations:

$$\frac{d^2w}{dx^2}+P\frac{dw}{dx}+Qw=0 \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

$$\frac{d^2v}{dx^2}+[(2/w)\frac{dw}{dx}+P]dv/dx=R/w \ldots\ldots\ldots\ldots(5).$$

w being supposed known from (4) gives

$$w^2\frac{dv}{dx}\varepsilon^{-\int Pdx}=A+\int wR\varepsilon^{\int Pdx}dx \ldots\ldots\ldots\ldots(6),$$

and $v=B+A\int\frac{dx}{w^2}\varepsilon^{-\int Pdx}+\int\frac{dx}{w^2}\varepsilon^{-\int Pdx}\int wR\varepsilon^{\int Pdx}dx \ldots\ldots\ldots\ldots(7).$

Now (4) is of the same form as (1) excepting that the right member is 0; so that if we have a solution of (4) we have that of (1) when $R=0$.

The given equation is

$$\frac{d^2y}{dx^2}-\frac{2x}{1+x^2}\frac{dy}{dx}+\frac{2}{1+x^2}y=0 \ldots\ldots\ldots\ldots(8);$$

then $P=-2x/(1+x^2)$, and a particular solution is

$y=x \ldots\ldots\ldots\ldots(9)$, or $w=x \ldots\ldots\ldots\ldots(10)$.

Then (7) gives $v=B--A\int\left(\frac{dx}{x^2}+1\right)=B+A(x-[x/1])$,

and $y=vx=Bx+A(x^2-1)$, the required solution.

[As will be seen from the last solution both forms are correct. The first form is given as the answer, on page 336 of *Byerly's Integral Calculus*. EDITOR.]

Also solved by *O. W. ANTHONY*, *W. C. M. BLACK*, *J. SCHEFFER*, *G. B. M. ZERR*. and *P. S. BERG*.

58. **Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in Columbian University, Washington, D. C.**

A line passes through a fixed point and rotates uniformly about this point. Another line passes through a point which moves uniformly along the arc of a given curve and rotates uniformly about this point. Develop a method for finding the locus of intersection of these two lines. Apply to case of circle and straight line.

Solution by the PROPOSER.

Let $y=f(x)$ be the equation of the curve, taking the fixed point as origin. Let (x_1, y_1) be the position of the moving point at time t. Call θ_1 the angle which the line through the fixed point originally makes with axis of x; also, let ω_1 be the rate of angular rotation. Then the equation of line is

$$y=\tan(\omega_1 t+\theta_1)x;$$

whence $t=(\omega_1/1)[\tan^{-1}(y/x)-\theta_1]$ ....(1).

Also the equation of other line is

$$y-y_1=\tan(\omega_2 t+\theta_2)(x-x_1);$$

whence $t=(\omega_2/1)\left(\tan^{-1}\dfrac{y-y_1}{x-x_1}-\theta_2\right)$ ....(2).

$$ds_1{}^2=dx_1{}^2+dy_1{}^2. \quad ds_1/dt=v_1. \quad \therefore\ v_1{}^2=[1+\overline{f'(x_1)}^2]^{\frac{1}{2}}\frac{dx_1{}^2}{dt^2}.$$

$$t=\frac{1}{v_1}\int[1+\overline{f'(x_1)}^2]^{\frac{1}{2}}dx \ldots\ldots(3). \quad y_1=f(x_1)\ldots\ldots(4).$$

To solve the problem, then, we integrate (3), solve the resulting equation for x_1, substitute this value in (4), and then substitute the values of x_1 and y_1 in (2), after which t is to be eliminated between the resulting equation and (1). To apply this method to the case of the straight line

$$y_1=f(x_1)=0. \quad x_1=v_1 t.$$

$\therefore$ From (2), $t=\dfrac{1}{\omega_1}\tan^{-1}\left(\dfrac{y}{x-v_1 t}\right)-\theta_1$.

Then $\dfrac{1}{\omega_1}\left(\tan^{-1}\dfrac{y}{x}-\theta_1\right)=\dfrac{1}{\omega_1}\left(\tan^{-1}\dfrac{y}{x-(v_1/\omega_1)(\tan^{-1}[y/x]-\theta_1)}-\theta_2\right)$.

Let $y=\rho\sin\phi$; $x=\rho\cos\phi$.

Then $\dfrac{1}{\omega_1}(\phi-\theta_1)=\dfrac{1}{\omega_2}\left[\tan^{-1}\left(\dfrac{\rho\sin\phi}{\rho\cos\phi-(v_1/\omega_1)(\phi-\theta_1)}\right)-\theta_2\right]$.

$$\therefore\ \frac{\rho\sin\phi}{\rho\cos\phi-(v_1/\omega_1)(\phi-\theta_1)}=\tan[\theta_2+\frac{\omega_2}{\omega_1}(\phi-\theta_1)].$$

$$\therefore\ \rho=\frac{(v_1/\omega_1)(\phi-\theta_1)\tan[\theta_2+(\omega_2/\omega_1)(\phi-\theta_1)]}{\cos\phi\tan[\theta_2+(\omega_2/\omega_1)(\phi-\theta_1)]-\sin\phi},$$

the polar equation of the curve.

The case of the circle leads to complicated results. The case of two fixed points is interesting. The equations of the intersecting straight lines may be written

$y=\tan(\omega_1 t+\theta_1)x$.............(1), and $y=\tan(\omega_2 t+\theta_2)(x-a)$............(2).

From (1) $\tan\phi=\tan(\omega_1 t+\theta_2)$.

$\therefore\ \phi=\omega_1 t+\theta_1.\quad t=(1/\omega_1)(\phi-\theta_1)$.

$\therefore\ \rho\sin\phi=\tan(\omega_2 t+\theta_2)(\rho\cos\phi-a)$;

whence $\rho=[a\tan(\omega_2 t+\theta_2)]/[\cos\phi\tan(\omega_2 t+\theta_2)-\sin\phi]$, or

$$\rho=\{a\tan\left[\frac{\omega_2}{\omega_1}(\phi-\theta_1)+\theta_2\right]\}/\{\cos\phi\tan\left[\frac{\omega_2}{\omega_1}(\phi-\theta_1)+\theta_2\right]-\sin\phi\},$$

$$=\{a\sin\left[\frac{\omega_2}{\omega_1}(\phi-\theta_1)+\theta_2\right]\}/\{\sin\left[\left(\frac{\omega_2}{\omega_1}-1\right)\phi+\theta_2-\frac{\omega_2}{\omega_1}\theta_1\right]\}\ \ldots\ldots\ldots\ldots\ldots\ldots(m).$$

If they both start from the initial horizontal position at the same time,

$$\theta_1=\theta_2=0,\ \text{and}\ \rho=a\frac{\sin[(\omega_2/\omega_1)\phi]}{\sin\{[(\omega_2/\omega_1)-1]\phi\}}.$$

A large number of curves is included in equation (m).

This problem was also solved by C. W. M. Black. His solution will appear in the next issue.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

41. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

If the earth were a perfect sphere and had a frictionless surface, what would be the motion of a particle placed at a given latitude ?

Solution by the PROPOSER.

Adopt as coördinates the latitude of the particle and the distance measured

in miles along a circle of latitude. Call the latitude λ, and the distance measured along the small circle x. Also let λ_1 be the initial latitude.

As the particle moves towards the equator under the resolved component of centrifugal force there will be no acceleration parallel to the equator.

Then clearly,

$$x=2\pi R(\cos\lambda-\cos\lambda_1)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Also for the acceleration towards the equator,

$$\frac{d^2(R\lambda)}{dt^2}=-\frac{2\pi^2}{T^2}R\sin2\lambda,\ \text{ or }\ \frac{d^2\lambda}{dt^2}=-\frac{2\pi^2}{T^2}\sin\lambda.$$

Let $d\lambda/dt=\mu$.

Then $\mu=\sqrt{2}\frac{\pi}{T}\sqrt{\cos2\lambda-\cos2\lambda_1}$.

Whence $t=\frac{1}{2}\sqrt{2}\frac{T}{\pi}\int_{\lambda_1}^{\lambda}\frac{d\lambda}{\sqrt{\cos2\lambda-\cos2\lambda_1}}=\frac{1}{2}\frac{T}{\pi}\int_{\lambda_1}^{\lambda}\frac{d\lambda}{\sqrt{\sin^2\lambda_1-\sin^2\lambda}}$.

Let $\sin\lambda=\sin\lambda_1\sin\phi$. Then

$$t=\tfrac{1}{2}\int_{\phi_1}^{\phi}\frac{d\phi}{\sqrt{1-\sin^2\lambda_1\sin^2\phi}}.\quad \phi_1=\tfrac{1}{2}\pi,\quad \phi=\sin^{-1}\left(\frac{\sin\lambda}{\sin\lambda_1}\right).$$

$$t=\tfrac{1}{2}\frac{T}{\pi}\left[\int_0^{\phi_1}\frac{d\phi}{\sqrt{1-\sin^2\lambda_1\sin^2\phi}}-\int_0^{\phi}\frac{d\phi}{\sqrt{1-\sin^2\lambda_1\sin^2\phi}}\right]$$

$$=\tfrac{1}{2}(T/\pi)[F(\sin\lambda_1,\ \tfrac{1}{2}\pi)-F(\sin\lambda_1,\ \phi)]\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Equation (1) gives the relation between the coördinates at any time, and (2) gives the time of motion.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

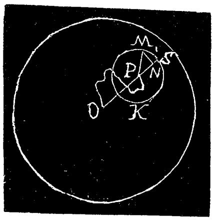
SOLUTIONS OF PROBLEMS.

39. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in Columbian University, Washington, D. C.

A man is at the center of a circular desert; he travels at a given rate but in a *perfectly* random manner. What is the probability that he will be off the desert in a given time ?

Solution by the PROPOSER.

Let R=the radius of desert, T=time, v=rate. Let P be the position of the man at any instant. Draw about P an infinitesimal circle, MSK. Call the



angle MPN, θ. Then $\theta=\cos^{-1}\dfrac{PN}{PM}$.

Now the rate at which the man must approach the circumference in order to be off in a given time is R/T. In an infinitely small time the distance will be $(R/T)dt$.

Also $PN=vdt$.

$$\therefore\ \theta=\cos^{-1}\left(\frac{R}{Tv}\right)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now R, T, and v are positive. Therefore the value of θ defined by equation (1) has to do with an angle less than $90°$.

Now if the man at each instant goes within the angle MPN, he will get off the desert in the given time. The chance that he will do this is

$$C=\frac{2\cos^{-1}[(R/Tv)]}{\pi}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Hence the required probability in given by (2).

If $R=0$, or T, $=\infty$, or $v=\infty$, $C=1$. If $R=Tv$, $C=0$.

If $R>Tv$, C is impossible.

Let $R=1$, and $v=1$. To find the time which he must have at his disposal in order that he may have half a chance to get off the desert.

Clearly $T=\sqrt{2}$.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

75. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

If 24 men, in 15 days of 12 hours each, dig a trench 300 yards long, 5 yards wide, 6 feet deep for 540 five-cent loaves when flour is \$8 a barrel; what is flour worth a barrel when 45 men, working $5\frac{1}{3}$ days of 10 hours each, dig a trench 125 yards long, 5 yards wide, 8 feet deep for 320 four-cent loaves? *Solve by proportion.*

76. Proposed by E. W. MORRELL, Professor of Mathematics in Montpelier Seminary, Montpelier, Vermont.

An eastern nobleman willed his entire estate to his three sons on the condition that the oldest should have one-half, the next one-third, and the youngest one-ninth. His estate, on inventory, was found to consist of 17 elephants. What should be the share of each ?

ALGEBRA.

78. Proposed by J. MARCUS BOORMAN, Consultative Mechanician, Counselor at Law, Inventor, Etc., Hewlett, Long Island, New York.

Solve $x^2+xy=10\ldots\ldots(1)$; $y^2+xy=15\ldots\ldots(2)$, for all the roots, and prove that they are the roots.

[Former solutions in print are defective. See *Analyst*, Vol. VIII, page 111; Vol. IX, page 53. J. M. B.]

79. Proposed by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

Of n persons A, B, C, etc., A first gives to the others as much as each of them already has; then B gives to the others as much as each then has; and so on for each in turn. Finally, A, B, C, etc., have respectively a, b, c, etc., dollars. How much had each at first?

80. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Solve $1+x^4=a(1+x)^4$.

GEOMETRY.

74. Proposed by ROBERT JUDSON ALEY, M. A., Professor of Mathematics in Indiana University, Fellow in Mathematics, University of Pennsylvania.

Let O be the center of the inscribed circle. AO produced meets the circumcircle in A'. Find the ratio of AO to OA'.

75. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, Ohio.

A plane passes through $(0, 0, c)$ and touches the circle $x^2+y^2=a^2$, $z=0$; determine the locus of the ultimate intersections of the plane.

76. Proposed by L. B. FRAKER, Bowling Green, Ohio.

Lines run from a point, P, within a triangular piece of land to the angles A, B, and C are 91, 102, and 80 rods, respectively; and a line 78 rods in length passing through the point, P, and terminating in the sides AC and BC cuts off 3024 square rods adjacent to angle C. Required the dimensions of the land.

CALCULUS.

65. Proposed by GEORGE LILLEY, Ph. D., LL. D., Portland, Oregon.

A string is wound spirally 100 times around a cone 100 feet high and 2 feet in diameter at the base. Through what distance will a duck swim in unwinding the string keeping it taut at all times, the cone standing on its base and at right angles to the surface of the water?

66. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

Around the top of a conical frustum,—base 5 feet, top 1 foot, altitude 100 feet,—is wound a rope 100 feet long and 1 inch thick. It is unwound by a hawk flying in one plane. How far does Mr. Hawk fly?

67. Proposed by BENJ. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, Ohio.

A man starts to walk at a uniform rate across a draw-bridge just as it begins to move. He walks the full length of the bridge and back, in the same time that it takes the bridge to make a half revolution. How far does he ride, the length of the bridge being 250 feet, and its velocity uniform about a center axis?

MECHANICS.

52. Proposed by **S. ELMER SLOCUM**, Union College, Schenectady, New York.

A chain 16 feet long is hung over a smooth pin with one end 2 feet higher than the other end and then let go. Show that the chain will run off the pin in about 7-5 second. [*Wright's Mechanics*, page 92.]

53. Proposed by **J. C. NAGLE, M. A., C. E.**, Professor of Civil Engineering, Agricultural and Mechanical College of Texas.

Find the locus of the center of gravity of an arc of constant length for a parabola.

54. Proposed by **C. H. WILSON**, Poughkeepsie, New York.

A body slides from rest down a series of smooth inclined planes, whose total heights are h feet. Show that the velocity at the bottom is $\sqrt{2gh}$ feet per second. [From *Wright's Mechanics*.]

AVERAGE AND PROBABILITY.

51. Proposed by **G. B. M. ZERR, A. M., Ph. D.**, Texarkana, Arkansas.

Three points are taken at random in a sphere and a plane passed through them. Find the average volume of the segment cut off from the sphere.

52. Proposed by **B. F. FINKEL, A. M., M. Sc.**, Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

A straight line of length a is divided into three parts by two points taken at random; find the chance that no part is greater than b. [From *Hall and Knight's Higher Algebra*.]

53. Proposed by **Samuel E. Harwood**, Professor of Mathematics, Southern Illinois State Normal University, Carbondale, Illinois.

Four Latin sentences are given. Number one has 12 words, two has 13 words, three and four have 6 each. What are the chances that two pupils will have them in the same order? Will the result vary with the number of pupils in the class?

EDITORIALS.

President George H. Harter, of Delaware College, Delaware, has just ordered a complete set of the MONTHLY.

We shall be pleased to pay 25 cents each for a limited number of copies of No. 6, Vol. I, and No. 11, Vol. II, of the MONTHLY. Any of our readers wishing to dispose of these numbers should write to us.

We are greatly pleased to note that the Board of City Trusts, Philadelphia, Pennsylvania, has recognized the long and faithful service of Professor Warren Holden in the following resolution: Resolved, That in consideration of forty-five years continued and faithful service, Warren Holden, A. M., Professor of Mathematics at Girard College, be retired January 31, 1897, at a salary of $2,500 per annum.

So far, we have received only a few letters respecting the matter of publishing the portraits of our contributors. We shall be pleased to hear still furthur, and those who favor the plan may send their photos to us at once.

The paper by the late Ansel N. Kellogg, of Chicago, published in this issue was sent to the MONTHLY at the suggestion of Professor Irving Stringham, of the University of California. Professor Stringham says, "They [the formulæ] take us back to methods that were in vogue at the beginning of the century. But they are much superior in accuracy and rapidity of convergence to any I have found in the older books. They will be of some interest, I think, to mathematical readers.

Their author, the late Ansel N. Kellogg, of Chicago, was for a number of years prominent in newspaper and business circles throughout the country. Though a very busy man, he found time for mathematical meditation, and that he could think efficiently in this domain the paper presented sufficiently attests."

As we are very anxious to increase the subscription to the MONTHLY we make the following liberal offers :

1. To any person sending us 75 new subscribers at our regular price, we will make a present of a handsome set of the *Century Dictionary and Encyclopedia.*

2. To any person sending us 50 new subscribers at our regular price, we will make a present of a $100 *Acme* or *Monarch Bicycle.*

3. To any person sending us 20 new subscribers at our regular price, we will make a present of the *Standard American Encyclopedia* [see advertisement on cover.]

4. To any person sending us 15 new subscribers at our regular price, we will make a present of a copy, in one volume, of the *Standard Dictionary* (Funk and Wagnalls').

In all cases the money must accompany the list of names sent in.

BOOKS AND PERIODICALS.

Determinants. Designed for High Schools, and Lower Classes of Colleges and Universities. By J. M. Taylor, M. S., Professor of Mathematics and Astronomy in the University of Washington and Director of the Observatory. 8vo. Cloth, 48 pages. Chicago : Werner School Book Company.

In this little book, Professor Taylor has set forth in a very clear and concise manner the fundamental principles of Determinants. We feel sure that this little work will go far towards popularizing the subject and bringing it within the easy comprehension of the students of our best High Schools. B. F. F.

Elements of Theoretical Physics. By Dr. C. Christiansen, Professor of Physics in the University of Copenhagen. Translated into English by W. F. Magie, Ph. D., Professor of Physics in Princeton University. Large 8vo. Cloth, 338 pages. Price, $3.25. New York: The Macmillan Co.

This work, at first sight, presents a formidable appearance in mathematical notation and formulæ, but by beginning with the introduction and carefully reading through it, the reader is led on to overcome difficulties by a force which can only be accounted for by the admirable, clear, and interesting treatment of the subjects. It presents the fundamental principles of Theoretical Physics and developes them so far as to bring the reader in touch with much of the new work that is now being done in that subject. It is not exhaustive in every respect, but is stimulating and informing and furnishes a view of the whole field, which will facilitate the reader's subsequent progress in special parts of it. The book is printed on good paper and is well bound. Its appearance could have been somewhat improved by not printing it so compactly. B. F. F.

Principles of Mechanism. A treatise on the Modification of Motion by Means of the Elementary Combinations of Mechanism or of the Parts of Machines. For use in College Classes, by Mechanical Engineers, etc., etc. By Stillman W. Robinson, C. E., D. Sc., till recently Professor of Mechanical Engineering in the Ohio State University. First Edition, first thousand. Large 8vo. Cloth, 309 pages. Price, $3.00. New York: John Wiley & Sons.

In this volume we have a thoroughly scientific treatise on mechanical movements. They are treated from the standpoint of both theory and practice. The work embodies the substance of lectures given by the author during the past twenty-seven years.

The work is largely addressed to those who are more conversant with the drawing board than with mathematics, so that the subject has been treated more from the standpoint of graphics than of pure analysis. This feature will popularize the work. The drawings, which are very suggestive, beautiful, and accurate, are very numerous. There are numerous reproductions from actual models. B. F. F.

(1) *Macaulay's Essay on Milton;* (2) *Shakespeare's Midsummer Night's Dream;* (3) *Scott's Woodstock;* (4) *Milton's L'Allegro, Il Penseroso, Comus, Lycidas;* (5) "*George Eliot's*" *Silas Marner.* Price of (1), (2), and (4) 20 cents, of (3) 60 cents, and of (5) 30 cents. American Book Company, New York, Cincinnati, and Chicago.

We notice collectively this group of texts from the "Eclectic English Classics" series, published by the American Book Company. The texts are well and carefully edited, with introductions and explanatory notes. (1), (3), and (5) have frontispiece portraits of John Milton, Oliver Cromwell, and "George Eliot", respectively. These books are clearly printed, the notes are concise and sufficient, and the introductions interesting and valuable. There is so much advantage in extending the use of these gems of English Literature in our schools, that a debt of gratitude is due the publishers for providing them in such serviceable shape and at a minimum cost. J. M. C.

An Elementary Treatise on Plane Trigonometry. By E. W. Hobson, Sc. D., and C. M. Jessop, M. A. 299 pages. Price, $1.25. Cambridge University Press. New York: Macmillan & Co.

This treatise on trigonometry is a work of recognized merit. The chapter on solution of trigonometrical equations is particularly full and valuable. The plan of the work is good and the execution thorough and satisfactory. J. M. C.

The Arena. An Illustrated Monthly Magazine. Edited by John Clark Redpath and Helen H. Gardner. Price, $3.00 per year in advance. Single numbers, 25 cents. Boston : The Arena Company.

The March number of *The Arena* is the initial issue of the magazine under the new management and editorship. The Company has been reorganized on a solid financial basis, and the current number of the magazine comes in a form and substance well calculated to win public favor, and following its well established policy of liberalism and reform.

The number opens with the first of a series of important contributions on the development and reform of city government in the United States. This first article is by the Hon. Josiah Quincy, Mayor of Boston, who therein expresses himself as in favor of the municipal ownership, though not necessarily the municipal operation, of public services, such as gas and electric lighting and street railways. An excellent portrait of Mayor Quincy forms the frontispiece to the number. The article by Professor LeConte, of the University of California, on "The Relation of Biology to Philosophy," is a searching adverse criticism of the seventh chapter of Professor Watson's recent work on "Comte, Mill, and Spencer;" but it is also very much more, being a thoroughly up-to-date exposition of the general theory of organic evolution, and its relation to religion as well as philosophy.

B. F. F.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single numbers, 25 cents. The Review of Reviews Co., 13 Astor Place, New York City.

The editor of the *Review of Reviews* comments in the March number on the Spanish program of reforms in Cuba, the United States Senate's attitude toward the arbitration treaty with England, the immigration bill, the proposed international monetary conference, President-elect McKinley's cabinet selections, the recent Senatorial elections, the New York Trust investigation, the famine situation in India, the affair of the Greeks in Crete, the foreign policy of Russia, the position of England, France, and the other great powers, and many other matters of current interest. B. F. F.

Errata in January Number.

On page 16, 2nd line of problem 55, for "grouhd" read *ground.*

On page 17, in the figure, join CF and CP.

On page 17, 1st line of solution II, for "$AMFHC$" read $AMFHC'$.

On page 20, line 1, complete the parenthesis after last term of equation.

On page 20, line 8, place — between two terms enclosed by brackets.

On page 20, line 14, for "$\frac{2}{2}\frac{1}{2}\pi^2a^4$" read $\frac{2}{3}\frac{1}{2}\pi^2a^4$.

On page 21, line 2, for "$\frac{16}{3}a^2$" read $\frac{16}{3}a^3$.

On page 21, line 8, read $d\rho=2a\cos^2\theta+a\cos\theta-a=0$.

On page 25, line 18, for "100" read 100th.

On page 25, line 25, read "add and subtract $B^2/4$, etc."

On page 26, line 15, for "$(2mp)^2$" read $(2mn)^2$.

On page 27, line 3, for "is" read in.

On page 28, line 15, for "$x++(x1)$" read $x+(x+1)$.

On page 28, line 20, for "2392" read 1393.

On page 32, lines 6, 11, and 12, read l where 1 occurs.

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. IV. MARCH, 1897. No. 3.

BIOGRAPHY.

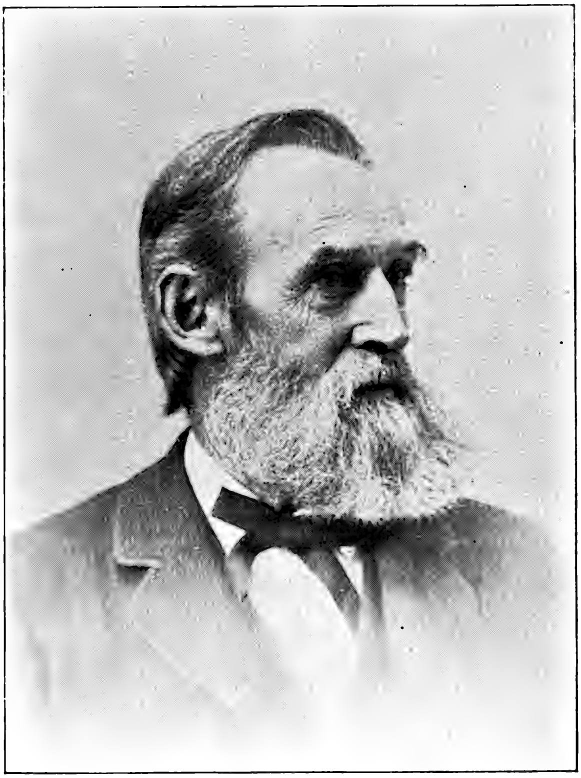
HUBERT ANSON NEWTON.

BY PROFESSOR ANDREW W. PHILLIPS.

HUBERT ANSON NEWTON was born in Sherburne New York, March 19, 1830, and died at New Haven on the 12th day of August, 1896.* He graduated from Yale, taking the degree of A. B., in 1850, and spent the next two and one-half years in mathematical study. He became tutor at Yale in 1853 and on account of the sickness and subsequent death of Professor Stanley, the whole work of the department of mathematics devolved upon him from the first. In 1855 his great ability was recognized in his election, at the early age of twenty-five, to a full professorship of mathematics at Yale, the duties of which he assumed after spending a year of study in Europe, where, under the inspiration of Chasles, he became especially interested in the subject of Modern Higher Geometry. He carried on most vigorously work and studies in various lines in addition to the duties of his professorship. Sometimes it was a profound study in pure Mathematics, sometimes a rich contribution to the education of the public, and sometimes an original investigation in the field of Astronomy.

He published in 1857 a paper on the Gyroscope in the *American Journal of Science*, and soon after, a paper in the *Mathematical Monthly*, in which he seems to have been the first to apply the principle of inversion in the solution of the problem of constructing circles tangent to three given circles. He showed how deeply rooted in his mind were the ideas of the Modern Geometry in his

*Professor Newton was Vice President of the American Mathematical Society at the time of his death. EDITOR.



HUBERT ANSON NEWTON.

elaborate papers published in the same journal in 1861 on the geometrical construction of certain curves by points, where he extended the ideas of Chasles, of de Jonquières, and of Poncelet. The subject of transcendental curves he studied for a long time with great interest, and constructed a myriad of interesting patterns, but contented himself with publishing, in the joint name of himself and his pupil, the discussion of the single group of equations which he found would give the most beautiful and symmetric forms, and which he had set for his pupil to investigate.

Professor Newton was very active in securing the prompt adoption of the Metric System of Weights and Measures, both by the Connecticut Legislature and by Congress after the Conference of Nations on the subject, held in Berlin in 1863. He wrote a popular tract in 1864, giving an explanation of the system. He contributed in 1865 to the Report of the House Committee on Weights and Measures at Washington, and also to the Report of the Smithsonian Institution on this subject. He prepared an appendix consisting of these tables in form for school instruction for one of the leading arithmetics, and interested the makers of scales rulers in graduating their devices for weighing and measuring according to the Metric System. He gave his ideas to the public freely in reference to the graphical representation of all sorts of statistical information, and contributed lavishly his ideas to the authors of mathematical books used in school and college class-rooms, although he published no text-books in his own name. He was the joint author with Professor Loomis of a most elaborate paper on the climate of New Haven, which was published in the Transactions of the Connecticut Academy of Arts and Sciences. He prepared articles on the subject of Meteors for two leading cyclopædias and contributed the mathematical and astronomical definitions to Webster's International Dictionary. Professor Newton was one of the highest authorities on the subject of Life Insurance and, besides the important actuarial work which he did, computed valuable tables published by the New York Insurance Department in 1868, and later in the New Englander, a paper on the Law of Mortality that prevailed among former members of the Yale Divinity School, and still later, in Professor Dexter's Annals and Biographies, on the Length of Life of the Early Yale Graduates.

But the contributions to human knowledge, which most entitle him to fame, are those which he made on the subject of meteors, shooting stars and comets. The facts of the great star shower of 1833 had given to two New Haven men—Professors Twining and Olmstead—a clue to the true theory of the shooting stars, and this, together with the interest which the men of science at Yale had kept up in the subject of meteors, influenced Professor Newton to direct his studies towards these bodies as the time drew near for a possible recurrence of the great November shower of 1833. In 1860 he published his first paper on this subject in the *Journal of Science*, entitled "The Fireball of November 15, 1859," and this was followed by two other papers in the same journal, one on the great fireball of August 10, 1861, in which also the August group of meteors was discussed; and the other on the two fireballs of August 2 and August

6, 1860. Professor Newton had gathered a large number of observations made by persons in the localities where these bodies had attracted attention, and treated the subject with special reference to determining their nature and their velocity. Early in 1863, at the request of the Connecticut Academy, he prepared a stellar chart suited to observations at all times, which was distributed to persons at various stations for observing the August meteors. A vast amount of material was thus collected for computing the altitudes of the meteors and for obtaining some idea of their velocities. In June, 1863, Professor Newton published in the *Journal* a paper on the "Evidence of the Cosmical Origin of Shooting Stars derived from the Dates of early Star Showers," which not only established beyond question the fact that the star showers are caused by the entrance into the earth's atmosphere of bodies revolving about the sun, but gave the key to the complete solution of the problem of the November meteors. In May, 1864, he published the original accounts of thirteen remarkable displays of the November shooting stars, ranging from A. D. 902 to 1833, and in July of the same year he published a second paper in which he derived from these accounts the length of the annual period, the length of the cycle, the mean motion along the ecliptic of the node of the orbit of the group, and the length of the part of the cycle during which showers may be expected. He also showed that there were only five possible periodic times which could satisfy the observed conditions, and of these the true orbit was probably either one with a period of 354.6 days or one with a period of 33.25 years. The first of these two he thought the more likely, and computed the other elements of that orbit, but he pointed out at the same time a criterion for determining which was the true orbit when the position of the radiant should be more accurately established.

In August, 1864, Professor Newton presented to the National Academy of Sciences a comprehensive memoir on the Sporadic Shooting Stars. He had shortly before this compiled a table of computed altitudes of certain shooting stars which included substantially all that had ever been published. Using this table as a basis, he deduced the distribution of meteor paths over the sky in altitude and in azimuth, the number of shooting stars that come into our atmosphere each day, the mean length of the visible part of the meteor paths, and the number of meteoroids in the space which the earth traverses. He also deduced the remarkable fact that the mean velocity could be determined from the number of shooting stars in the different hours of the night.

These papers of Professor Newton aroused the greatest interest among mathematicians and astronomers in the subject of meteors, and especially in the star showers predicted for November, 1865 and 1866. The facts of these showers confirmed to a remarkable degree Professor Newton's theories. Leverrier and Schiaparelli, however, by independent methods showed that the period of the group was most probably 33.25 years, and Professor Adams, in 1867, by applying Professor Newton's criterion added the last link in establishing this as the true orbit of the November meteoroids.

Professor Newton, by his papers of 1863 and 1864, laid the foundation of

the Science of Meteoric Astronomy. His subsequent papers, nearly thirty in number, cover almost every topic connected with the subject. Whether in his reviews of the facts concerning the November shooting stars in the successive years from 1864 to 1869, or in the discussion of the Biela meteors of 1872 and of 1885, or in his treatment of such topics as the origin of comets, or the direct motion of comets of short period, the capture of comets by Jupiter, the effect upon the earth's velocity produced by small bodies entering the asmosphere, the relation to the earth's orbit of the former orbits of those meteorites in our collections, which were seen to fall, one prominent characteristic of his investigation was always its exhaustive character. For, whatever Professor Newton, did it was not worth the while of any one else to cover the same field.

Besides the papers which he published, his scientific activities outside the duties of his professorship were numerous and important. He organized a mathematical society in the early '60s to which he was the principal contributor, and to the successor of this society, the Yale Mathematical Club, organized in 1887, he contributed more than a score of papers. He was for many years a member of the Publishing Committee of the Connecticut Academy of Arts and Sciences. He was an associate editor of the *American Journal of Science* for thirty years. He was one of the principal founders of the Yale Observatory and practically its director till near the time of his death.

The appreciation in which his scientific ability and his labors were held is shown in the honors which he received. In 1862 he was made a member of the American Academy of Arts and Science. He was one of the original charter members of the National Academy of Sciences, founded in 1863. In 1867 he was made a member of the American Philosophical Society of Philadelphia. The degree of LL. D. was conferred upon him by Michigan University in 1868. He was made an Associate of the Royal Astronomical Society in 1872. He was Vice President of the American Association for the Advancement of Science, presiding over the section of Mathematics and Astronomy in 1875, and was President of the Association in 1885. He was made a Foreign Honorary Fellow of the Royal Society of Edinburgh in 1886, and a Foreign Member of the Royal Society of London in 1892.

At the April meeting of the National Academy in 1888 the value of Professor Newton's scientific work was publicly recognized by that body, in awarding to him the J. Lawrence Smith gold medal for his contributions to Meteoric Astronomy. His reply to the address of presentation reveals at once his modesty and his own true scientific spirit.

"Sir: I beg to express to the Academy my high appreciation of the honor you have conferred upon me. To discover some new truth in nature, even though it concerns the small things in the world, gives one of the purest pleasures in human experience. It gives joy to tell others of the treasure found. When, therefore, those best able to judge of the value of this addition to human knowledge say that it is worthy of their special public commendation, that joy is greatly increased.

I shall cherish this memorial also for that it bears the likeness of one whose true scientific spirit we all learned to admire, and whom, for his genial character, we all learned to love.''

The achievements of Professor Newton, great as they were from a scientific standpoint, give no adequate idea, taken in themselves, of his power and influence. These, in a larger sense have become a part of the organic life of the University where his work was done. He built up, during a leadership of forty years, a strong and symmetrical department of Mathematics, by his comprehensive grasp of the trend of Mathematical thought, and by his wonderful power of divining the paths which lead out to fruitful fields of research, both within the domain of pure mathematics and in its applications to other sciences. Nor was the best part of his academic activities merely in his own department of studies. In moulding the general policy of the institution his counsel was invaluable ; in establishing and maintaining the moral and intellectual standards, his influence was preëminent ; the University bears the indelible impress of a life consecrated to the development of the noblest ideals.

Yale University.

ON THE SOLUTION OF THE QUADRATIC EQUATION.

By G. A. MILLER, Ph. D., Paris, France.

[Continued from January Number.]

The solution of the quadratic equation

$$a_0x^2+a_1x+a_2=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad A$$

is clearly equivalent to finding the two factors which are linear in x of the quantic

$$a_0x^2+a_1x+a_2.$$

When we ask whether this quantic has linear factors it is necessary to consider the domain of rationality to which we confine our attention. For illustration, we may consider the special quantic

$$x^2-4x+1.$$

If we confine ourselves to the simplest domain of rationality, viz : the domain which consists of all the rational numbers, we have to say that this quantic has no linear factors. In other words, it is irreducible in this domain. Howev-

er, if we enlarge this domain by adding to it the irrational number $\sqrt{3}$* we obtain a domain in which the quantic is clearly reducible. This domain is composed of all the numbers whose form is

$$\alpha+\beta\sqrt{3} \qquad (\alpha \text{ and } \beta \text{ being any rational numbers}).$$

According to the fundamental theorem of algebra a quantic which involves only a single variable can always be resolved into its linear factors in the domain obtained by enlarging the domain of its coefficients, if necessary, so as to include suitable new numbers. If the coefficients lie in the domain of the complex numbers the added numbers must also lie in this domain. If a quantic involves several variables it may remain irreducible even when the domain is enlarged in every possible manner.

Let x_1 and x_2 be the two roots of A. Since every rational symmetric function of the roots of an algebraic equation can be expressed rationallly in terms of its coefficients we know the value of any rational symmetric function of x_1 and x_2. This value must lie in the domain of the coefficients. In particular, we know the value of any even power of x_1-x_2. The value of the square is given by the equation

$$(x_1-x_2)^2=x_1{}^2+x_2{}^2-2x_1x_2=(x_1+x_2)^2-4x_1x_2=a_1{}^2-4a_0a_2/a_0{}^2.$$

To find the difference of the roots from the last equation we have to extract the square root of the last member. This may be impossible in the domain of the coefficients. If this domain forms a group with respect to the extraction of the square root it is clearly possible in this domain. We know that the system of ordinary complex numbers forms a group with respect to the extraction of any root. Hence we see that, if a_0, a_1, a_2 lie in the domain formed by the ordinary complex numbers, the difference of the roots of A as well as the sum of these roots must lie in the same domain.

The roots themselves may be found from these two functions by means of addition and subtraction. As any domain includes all the quantities resulting by applying these operations to any of its quantities the roots of A must also lie in the given domain of rationality. The roots may also be found by observing that their general linear function

$$\alpha x_1+\beta x_2$$

is rationally expressible as follows :†

$$\alpha x_1+\beta x_2\equiv\tfrac{1}{2}(\alpha+\beta)(x_1+x_2)+\tfrac{1}{2}(\alpha-\beta)(x_1-x_2)$$

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*By enlarging a domain of rationality by the addition of a quantity is meant the forming of the smallest domain that contains the given domain and the added quantity.

†This is an illustration of the general theorem that any rational function of the roots of an algebraic equation of degree n is rationally expressible in terms of a $n!$ valued function of the n roots.

$$\equiv \tfrac{1}{2}-(\alpha+\beta)(a_1/a_0)+(\alpha+\beta)/2a_0\sqrt{a_1{}^2-4a_0a_2}.$$

By letting $\alpha=1$, $\beta=0$, and $\alpha=0$, $\beta=1$ in this identity we obtain the values of x_1 and x_2 respectively.

As the ordinary complex numbers do not only form a domain of rationality but also a group* with respect to what is frequently called the most general algebraic operation, viz : that represented by

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots\ldots a_n=0$$

(a_0, a_1, a_2, ..., a_n being ordinary complex numbers and n any posiive integer), and as they obey the commutative, distributive and associative laws of operation just like real numbers and also the law that a product cannot be zero unless one of the factors is zero, it is clear that we can reason quite generally in regard to symbols representing such numbers. It is probably largely due to this fact that other number systems are not more generally employed. In fact, no really different number system was developed until 1843. In this year Sir William Hamilton discovered and communicated to the Royal Irish Academy the system known as *Quaternions*, which is perhaps still the most important system besides that of the ordinary complex numbers. In the following year Grassmann published his *Ausdehnungslehren* in which he used a number system of a somewhat different form.

Among the investigations of later years those of Weierstrass have probably received the most attention† although important developments have been made in other directions. The fact that the ordinary complex numbers correspond to the points of a plane very naturally led to the thought that a system of higher complex numbers of the form

$$\alpha+\beta i+\gamma j \qquad (\alpha,\ \beta,\ \gamma \text{ being any real numbers})$$

might correspond to the points of space. It was easy to show that the product of two such numbers, multiplied according to the rules of ordinary numbers, may be zero when neither of the factors is zero.‡ This result naturally led to the study of numbers which do not obey all the laws of operations which the ordinary numbers obey.

The main purpose of the preceding remarks was to obtain a fairly clear view of number and of the domain of rationality as these two concepts are fundamental in the study of the solution of algebraic equations. Incidentally we indicated several methods of solving the quadratic equation A. We proceed now to consider some of the other methods of solving this equation. We shall not aim at a complete enumeration of the methods by which A may be solved. In

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*It seems that Poincara was the first who considered the general number systems directly as groups. Cf. Comptes Rendus, t. 99, p. 740.

†*Gottinger Nachrichten*, 1884, page 395.

‡Cf. Harkness and Morley, *Theory of Functions*, page 8.

fact, if we would consider each modification of the operations of finding the roots of A as a new method the number of these methods would clearly be infinite. We may, for instance, form an infinite number of quantics of the form of a quadratic each of which contains the first member of A as a factor. For A may be written in the form

$$a_0x^2+a^2=a_1x.$$

Squaring both members and combining we have

$$ax^4+bx^2+c=0,$$

(a, b, c belonging to the same domain as a_0, a_1, a_2). Since the result is of the same form as A we may repeat the operation any number of times. Hence A is a factor of the quantic

$$A_0x^{2^\alpha}+A_1x^{2^{\alpha-1}}+A_2,$$

(A_0, A_1, A_2 belonging the same domain as a_0, a_1, a_2 and α being any positive integer). The roots of any one of the equations obtained by making these quantics equal zero include the roots of A. As the roots of

$$A_0y^2+A_1y+A_2=0$$

are the $2^{\alpha-1}$ powers of the roots of A it is clear that none of these transformations can simplify the solution of A. By elimination we may clearly obtain an indefinite number of additional equations containing the roots of A from the given system. In particular, if we eliminate the constant from the biquadratic equation by means of A we obtain a biquadratic equation which has the roots of A and two zero roots. Upon this elimination depends a solution recently publish- in this journal. The same result might be obtained by multiplying both members of A by x^2. It is, in general, not well to raise the degree of A in the process of solution since this introduces additional roots and therefore makes the operation more complex.

Perhaps the best known method of solving A is that by which its first member is made a perfect square by the addition of the same quantity to each member. To make the quantic

$$a_0x^2+a_1x+a_2$$

a perfect square without altering its degree we may add to it the quantic

$$ax^2+bx+c$$

where two of the three numbers a, b, c are entirely arbitrary since it is only necessary that the discriminant vanishes. This idea is frequently expressed by say-

ing that the quantic to be added can be chosen in a doubly infinite number of ways. Since this quantic must also be a perfect square its own discriminant must also vanish. As this imposes another condition on its coefficients we can select the trinomial to be added to both members of A in only a simply infinite number of ways.

This number of choices might at first appear too small since in the ordinary method by which we add a constant to both members of A we apparently select both a and b arbitrarily since we let both equal zero. This would imply a doubly infinite number of choices. This apparent contradiction is explained by the fact that the vanishing of the discriminant of the added trinomial, i. e., the equation

$$b^2=4ac$$

indicates that at least two of the coefficients, including b, must be zero when one is zero. Hence the ordinary method implies that one of the coefficients of the added trinomial is selected arbitrarily and the other in accord with this equation.

To illustrate we inquire what quantics may be added to both members of the special equation

$$x^2-4x+1=0$$

so as to make both members perfect squares. Adding the given general quantic we have the equations

$$(a+1)x^2+(b-4)x+c+1=ax^2+bx+c.$$

Since the discriminants of both members must vanish we have

$$(b-4)^2=4(a+1)(c+1) \text{ and } b^2=4ac.$$

If we assign to b the arbitrary number 2 and eliminate c we have

$$a^2+a+1=0.$$

Hence a and c are the imaginary cube roots of unity, ω_1 and ω_2, and the given equation becomes*

$$-\omega_1{}^2x^2-2x-\omega_2{}^2=\omega_1x^2+2x+\omega_2$$

or

$$-1(\omega_1{}^2x^2+2x+\omega_2{}^2)=\omega_2{}^2x^2+2x+\omega_1{}^2.$$

Extracting the square root from both members we have

$$\pm i(\omega_1x+\omega_2)=\omega_2x+\omega_1$$

*It should be observed that the product of the two imaginary cube roots of unity is unity and that the square of one is equal to the other.

or
$$x=\frac{\omega_1 \mp i\omega_2}{\pm i\omega_1-\omega_2}=2\pm\sqrt{3}.$$

If we let $b=4$ the first discriminant shows that one of the two factors $a+1$, $c+1$ must vanish. If we suppose that the former vanishes the given equation becomes

$$-3=-x^2+4x-4 \text{ or } x^2-4x+4=3.$$

If we suppose that the latter of the given factors vanish we obtain the equation
$$4x^2-4x+1=3x^2.$$

Instead of assigning an arbitrary value to b we might clearly assign an arbitrary value to either of the other coefficients. The simplest method is probably that in which a is made equal to zero. By making a and b equal to the corresponding coefficients with the signs changed of the equation which is to be solved and selecting c so as to satisfy the equation.

$$b^2=4ac$$

we have another simple rule for completing the square. A number of other fairly convenient rules can easily be derived from what precedes.

That we can assign the given values to a and b follows from the first of the given discriminants. If we assign this value to a we determine the value of b at the same time but if we commence by assigning the given value to b neither a nor c are fully determined. We still say that the number of choices is simply infinite since a finite number multiplied into a simply infinite number is said to give a simply infinite product. The preceding remarks apply evidently also to the slight modification of the given method which consists in writing A in the form
$$a^2-b^2=0 \text{ instead of } a^2=b^2$$

and factoring the first member according to the well known formula

$$a^2-b^2=(a+b)(a-b)=(-b-a)(b-a)$$

instead of extracting the square root of the two members.

Another simple method of solving A may be described as follows: The equation A is satisfied by the affixes of two points and gives the elementary symmetric functions of these affixes. As all rational symmetric functions can be expressed rationally in terms of the elementary symmetric functions we know the affix of the middle point of the join of the roots. If the points of the plane are so transformed that this point becomes the origin the roots are the affixes of the extremities of a diameter of a circle whose center is the origin. Hence the equa-

tion in the new variable must be a pure quadratic and the solution is readily completed. If we do not assume that the coefficients are real, one root may be real while the other is imaginary. In fact the roots may be the affixes of any two points.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M. (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from January Number.]

PROPOSITION XXV. *If two straights (Fig. 30.) AX, BX existing in the same plane (standing upon AB, one indeed at an acute angle in the point A, and the other perpendicular at the point B) so always approach more to each other mutually, toward the parts of the point X, that nevertheless their distance is always greater than a certain assigned length, the hypothesis of acute angle is destroyed.*

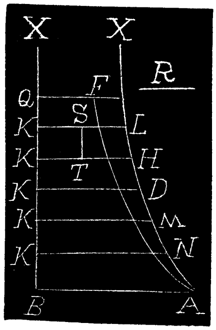


Fig. 30.

PROOF. Let R be the assigned length. If therefore in BX is assumed a certain BK any chosen multiple of the proposed length R; it follows (from the preceding Scholion) that the perpendicular erected from the point K toward the parts of AX will meet it at some point L; and again (from the present hypothesis) it follows that this KL will be greater than the aforesaid length R. Furthermore BK is understood divided into portions KK, each equal to R, even until KB is itself equal to the length R. Finally from the points K are erected to BX perpendiculars meeting AX in points L, H, D, M, even to the point N nearest the point A. Now I proceed thus.

The four angles together of the quadrilateral $KHLK$, more remote from the base AB, will be (from the preceding Proposition) greater than the four angles together of the quadrilateral $KDHK$, nearer to this base; of which quadrilateral in the same way the four angles together will be greater than the four angles together of the quadrilateral $KMDK$ subsequent toward this base. And so always even to the last quadrilateral $KNAB$, whose four angles together assuredly will be the least, in reference to the four angles together of each of the quadrilaterals ascending toward the points X.

But since are present as many quadrilaterals described in the aforesaid manner, as are, except the base AB, perpendiculars let fall from points of AX to

the straight BX; the sum of all the angles together, which are comprehended in these quadrilaterals can be reckoned. We assume that there are nine such perpendiculars let fall, and therefore so nine quadrilaterals.

We get (from Eu. I. 13) as equal to four rights the angles comprehended hither and yon at the two points of those eight perpendiculars, which lie in the middle between the base AB and the more remote perpendicular LK. So the sum of all these angles will be 32 rights.

There remain two angles at the perpendicular LK, and two at the base AB. But the angles one indeed at the point K and the other at the point B are supposed right; but the angle at the point L (from the Cor. after P. XXIII.) is obtuse. Wherefore (even neglecting the acute angle at the point A) the sum of all the angles which are comprehended by these nine quadrilaterals exceeds 35 rights. But hence follows, that the four angles together of the quadrilateral $KHLK$, more remote from the base lack less from four rights than the ninth part of one right; and that indeed even if an equal portion of the aforesaid sum of all the angles pertained to each of those quadrilaterals.

Therefore less yet will be the entered defect, since the sum of the four angles together of this quadrilateral $KHLK$ was shown the greatest of all, in relation to the four angles together of the remaining quadrilaterals.

But again; in consequence of the supposition upon which this proposition proceeds, so great a length of BK can be assumed, that as many quadrilaterals as we choose may be made on bases KK, each equal to the assigned length R.

Wherefore the defect of the four angles together of this more remote quadrilateral $KHLK$ from four rights is shown ever less both than a hundredth and than a thousandth, and thus under any assignable part of a right. Further however, LK and HK will be always (in accordance with the aforesaid supposition) greater than the designated length R. Therefore if in KL and KH are assumed KS and KT equal to KK or the length R; ST being joined, the two angles together KST, KTS will be greater, in hypothesis of acute angle, than the two angles together (from Cor. after P. XVI.) at the points H and L in the quadrilateral $THLS$, or the quadrilateral $KHLK$; and therefore (the common right angles at the points K, K being added) the four angles together of the quadrilateral $KTSK$ will be greater than the four angles together of that quadrilateral $KHLK$.

But now, since on one hand is stable and given the quadrilateral $KTSK$, in as much as constant in the given base KK, which indeed is taken equal to the assigned length R, and again constant in the two perpendiculars TK, SK equal to this base, and finally in the joining TS, which comes out completely determinate; and on the other hand the four angles together of this stable and given quadrilateral have now been shown greater than the four angles together of the quadrilateral $KHLK$ distant as far as we choose from the base AB; assuredly it follows, that the four angles together of this stable and given quadrilateral $KTSK$ are greater than any sum of angles, which lacks however you choose of being four right angles; since already it has been shown that a quadrilateral $KHLK$ can always be designated such that its four angles together shall fall short of four

rights less than any assignable part of a right. Therefore the four angles together of this stable and given quadrilateral either are equal to four rights or greater.

But then (from P. XVI.) is established the hypothesis either of right angle or of obtuse angle ; and therefore (from P. V. and P. VI.) the hypothesis of acute angle is destroyed.

So is established that the hypothesis of acute angle will be destroyed, if two straights existing in the same plane so approach each other mutually ever more, that nevertheless their distance is always greater than any assigned length.

Hoc antem erat demonstrandum.

COROLLARY I. But (the hypothesis of acute angle destroyed) the controverted Pronunciatum Euclidaeum is manifest from P. 13 of this ; just as that by me in this place it would be disclosed I promised in Scholion III after P. XXI of this, where we spoke of the attempt of the Arab Nassaradin.

COROLLARY II. On the other hand from this proposition, and from the preceding XXIII is manifestly gathered that not sufficient for establishing Euclidean geometry are two following points. One is : that we designate by the name of parallels those straights, which existing in the same plane possess a common perpendicular. The second indeed ; that all straights existing in the same plane, of which there is no common perpendicular, and therefore which according to the assumed definition are not parallel, must, being produced toward either part ever more, somewhere meet each other, if not at a finite, at least at an infinite distance.

For again it would be requisite to demonstrate, that any two straights existing in the same plane, upon which a certain straight cutting makes two internal angles toward the same parts less than two right angles, nowhere else can receive a common perpendicular.

But that, this demonstrated, Euclidean geometry is most exactly established, will be shown below.

[To be Continued.]

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from January Number.]

XXVII. Let ABC be a triangle, right-angled at C. With O, the middle of AB, as a center, describe a circle to which either of the other sides, as BC, shall be tangent. Then,

$$BD \cdot BE = \overline{BP}^2;$$

or $(\frac{1}{2}c-\frac{1}{2}b)(\frac{1}{2}c+\frac{1}{2}b)=\frac{1}{4}a^2$. $\therefore c^2=a^2+b^2$.

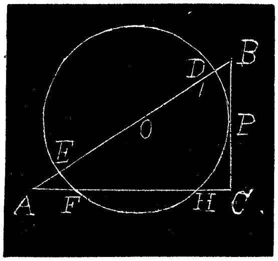


Fig. 21.

This and XVI are special cases of a more general form. For O may be any point in AB, such that the ratio of OB to AB shall be n. Our equation would then become $(nc-nb)(nc+nb)=n^2a^2$; whence, $c^2=a^2+b^2$.

XXVIII. Fig. 21.

Suppose $BC<AC$. Then $HC \cdot FC=\overline{PC}^2$.

But $HC \cdot FC=AF \cdot AH=AE \cdot AD=(\frac{1}{2}c-\frac{1}{2}b)(\frac{1}{2}c+\frac{1}{2}b)$, and $\overline{PC}^2=\frac{1}{4}a^2$.

$\therefore c^2=a^2+b^2$.

From $BC<AC$ pass to $BC=AC$ by the theory of limits.

XXIX. Let ABC be a triangle, right-angled at C. Describe a circle, such that its center O shall be in AB, and to which the other sides shall be tangent.

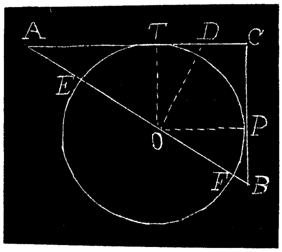


Fig. 22.

Draw OD perpendicular to AB. Then,

$$\overline{AT}^2=AE \cdot AF=\overline{AO}^2-\overline{EO}^2=\overline{AO}^2-\overline{TC}^2 \quad \ldots\ldots\ldots\ldots (1).$$

$$\overline{BP}^2=BF \cdot BE=\overline{BO}^2-\overline{FO}^2=\overline{BO}^2-\overline{CP}^2 \quad \ldots\ldots\ldots\ldots (2).$$

Now, $AO : OT :: AD : OD$;

$\therefore AO \cdot OD=OT \cdot AD$.

And, since $OD=OB$, $OT=TC=CP$, and $AD=AT+TD=AT+BP$,

$$\therefore AT \cdot TC+CP \cdot BP=AO \cdot OB \quad \ldots\ldots\ldots\ldots (3).$$

Adding (1), (2), and $2\times(3)$,

$$\overline{AT}^2+\overline{BP}^2+2AT \cdot TC+2CP \cdot BP=\overline{AO}^2-\overline{TC}^2+\overline{BO}^2-\overline{CP}^2+2AO \cdot OB;$$

$$\therefore \overline{AT}^2+2AT \cdot TC+\overline{TC}^2+\overline{BP}^2+2BP \cdot CP+\overline{CP}^2=\overline{AO}^2+2AO \cdot OB+\overline{BO}^2;$$

$$\therefore \overline{AC}^2+\overline{BC}^2=\overline{AB}^2.$$

XXX. Let ABC be a triangle, right-angled at C. Describe a circle, such that its center O shall be in one of the legs, as AC, and to which the other leg

and hypotenuse shall be tangent.

Then $\overline{AD}^2 = AE \cdot AC = \overline{AC}^2 - 2AC \cdot OE$;

and $\overline{BD}^2 = \overline{BC}^2$;

$\therefore$ Adding, $\overline{AD}^2 + \overline{BD}^2 = \overline{AC}^2 + \overline{BC}^2 - 2AC \cdot OE$.

$\therefore$ $\overline{AD}^2 + 2AC \cdot OE + \overline{BD}^2 = \overline{AC}^2 + \overline{BC}^2$.

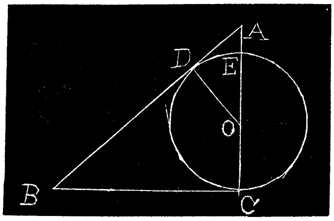


Fig. 23.

Now $AC : AC :: OD(=OE) : BC(=BD)$;

$\therefore$ $AD \cdot BD = AC \cdot OE$; $\therefore$ $\overline{AD}^2 + 2AD \cdot BD + \overline{BD}^2 = \overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

XXXI. Fig. 23.

$\overline{AD}^2 (=(AB - BD)^2) = \overline{AC}^2 - 2AC \cdot OE$,

$\therefore$ $\overline{AB}^2 - 2AB \cdot BD + \overline{BD}^2 = \overline{AC}^2 - 2AC \cdot OE$.

Adding $\overline{BD}^2 = \overline{BC}^2$, $\overline{AB}^2 - 2BD \cdot AD = \overline{AC}^2 + \overline{BC}^2 - 2AC \cdot OE$.

$\therefore$ $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

XXXII. Let ABC be a triangle right-angled at C. Draw AE parallel to BC and $=AC$. With O, the middle of AE, as a center, describe a circle, to which both AC and BC shall be tangent. Then,

$\overline{BT}^2 (=(a-b)^2) = BD \cdot BA = c(c - AD)$ .... (1).

Also, $AD : a :: 2b : c$.

$\therefore$ $AD = 2ab/c$ .... (2).

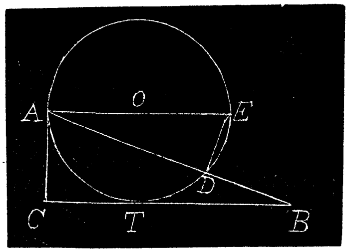


Fig. 24.

(2) in (1), $(a-b)^2 = c^2 - 2ab$. $\therefore$ $c^2 = a^2 + b^2$.

From unequal sides about the right angle pass to equal sides by the theory of limits.

[To be Continued.]

TWO DEVELOPMENTS.

By E. D. ROE, JR., Associate Professor of Mathematics in Oberlin College, Oberlin, Ohio.

[A paper read at the January meeting of the American Mathematical Society.]

It is desired to call attention here to two developments, whose statement and discussion the writer has no where met, except for $n=2$, and then not from the point of view to be suggested here. Yet it is quite possible that they may be found elsewhere.*

I. Formulas.

If $u=f(x_1, x_2, \ldots\ldots x_n)$ is a function of n variables, Δu will be used to denote the total increment in the function due to a change in all the variables, while $\Delta_{x_1, x_2 \ldots\ldots x_r} u$ will denote an increment due to a change in r variables. The two developments are as follows:

$$1.\quad u+\Delta u=u+\Delta_{x_1}u+\Delta_{x_2}u+\ldots\ldots\Delta_{x_n}u$$
$$+\Delta_{x_1}\Delta_{x_2}u+\Delta_{x_1}\Delta_{x_3}u+\ldots\ldots\Delta_{x_1}\Delta_{x_2}\ldots\ldots\Delta_{x_n}u$$

$$=u+\sum_{r=1}^{r=n}{}_r\ \sum_{\alpha_1=1}^{\alpha_1=n-r+1}{}_{\alpha_1}\ \sum_{\alpha_2=\alpha_1+1}^{\alpha_2=n-r+2}{}_{\alpha_2}\ \ldots\ldots$$
$$\sum_{\alpha_{r-1}=\alpha_{r-2}+1}^{\alpha_{r-1}=n-1}{}_{\alpha_{r-1}}\ \sum_{\alpha_r=\alpha_{r-1}+1}^{\alpha_r=n}{}_{\alpha_r}\Delta_{x_{\alpha_1}}\Delta_{x_{\alpha_2}}\ldots\ldots\Delta_{x_{\alpha_r}}u.$$

Or in determinant notation,

$$\begin{vmatrix} \Delta & 1 \\ -1 & 1 \end{vmatrix} u = \begin{vmatrix} \Delta_{x_1} & 1 & 1\ldots\ldots\ldots 1 & 1 \\ -1 & \Delta_{x_2} & 1\ldots\ldots\ldots 1 & 1 \\ -1 & -1 & \cdot\ \cdot\qquad\quad \vdots & \vdots \\ \ldots\ldots\ldots & & \cdot\ .\Delta_{x_n} & 1 \\ -1 & -1 & \ldots\ldots\ -1 & 1 \end{vmatrix} u$$

*Since the above was read Professor Fiske of Columbia, has informed the writer that "Dr. McClintock called attention to the fact, that the first result is contained in a general formula which he gave in Vol. II. page 116 of the *American Journal of Mathematics*. His formula (77) reduces in a special case to

$$\phi(E)=\phi(E_{x_1}E_{x_2}\ldots\ldots E_{x_n}) \text{ where } E=1+\Delta.$$

His operation S reduces to E when

$$\Psi=1."$$

$$=(1+\Delta_{x_1})(1+\Delta_{x_2})\ldots\ldots(1+\Delta_{x_n}u)=\overset{r=n}{\underset{r=1}{\pi}}r(1+\Delta_{x_r})u,$$

so that as operators $(1+\Delta)=\overset{r=n}{\underset{r=1}{\pi}}r(1+\Delta_{x_r})$, or the operator $1+\Delta$ is developable into the product of n operators.

2. $$u-\int du=u-\int d_{x_1}u-\int d_{x_2}u\ldots\ldots-\int d_{x_n}u+\int\int d_{x_1}d_{x_2}u+\int\int d_{x_1}d_{x_3}u+$$

$$\ldots\ldots+(-1)^n\int\int\ldots\ldots\int d_{x_1}d_{x_2}\ldots\ldots d_{x_n}u$$

$$=u+\sum_{r=1}^{r=n}r\sum_{\alpha_1=1}^{\alpha_1=n-r+1}\alpha_1\sum_{\alpha_2=\alpha_1+1}^{\alpha_2=n-r+2}\alpha_2\ldots\ldots$$

$$\sum_{\alpha_r=\alpha_{r-1}+1}^{\alpha_r=n}\alpha_r(-1)^r\int\int\ldots\ldots\int d_{x_{\alpha_1}}d_{x_{\alpha_2}}\ldots\ldots d_{x_{\alpha_r}}u.$$

Or in determinant notation,

$$\begin{vmatrix}-\int d & 1\\ -1 & 1\end{vmatrix}u=\begin{vmatrix}-\int d_{x_1} & 1 & 1\ldots\ldots 1 & 1\\ -1 & -\int d_{x_2} & 1\ldots\ldots 1 & 1\\ -1 & -1 & \vdots & \vdots\\ \ldots\ldots & \ldots & -\int d_{x_n} & 1\\ -1 & -1\ldots\ldots & -1 & 1\end{vmatrix}u$$

$$=(1-\int d_{x_1})(1-\int d_{x_2})\ldots\ldots(1-\int d_{x_n})u=\overset{r=n}{\underset{r=1}{\pi}}r(1-\int d_{x_r})u=0,$$

so that as operators $(1-\int d)=\overset{r=n}{\underset{r=1}{\pi}}r(1-\int d_{x_r})$, or the operator $1-\int d$ is developable into the product of n operators.

II. Proofs.

1. Let $u=f(x_1)$, then $\Delta u=\Delta_{x_1}u$, and $u+\Delta n=u+\Delta_{x_1}u=(1+\Delta_{x_1})u$.

Let $u=f(x_1x_2)$, then by the preceding, $u+\Delta_{x_2}u=(1+\Delta_{x_2})u$.

Apply the operator $1+\Delta_{x_1}$ to this equation, and

$$u+\Delta_{x_2}u+\Delta_{x_1}u+\Delta_{x_1}\Delta_{x_2}u=(1+\Delta_{x_1})(1+\Delta_{x_2})u.$$

By writing out the left member,

$$f(x_1,x_2)+f(x_1,x_2+\Delta_{x_2})-f(x_1,x_2)+f(x_1+\Delta x_1,x_2)-f(x_1x_2)+f(x_1+\Delta x_1,x_2+\Delta x_2)$$
$$-f(x_1,\ x_2+\Delta x_2)-f(x_1+\Delta x_1,\ x_2)+f(x_1,\ x_2)$$
$$=f(x_1x_2)+f(x_1+\Delta x_1,\ x_2+\Delta x_2)-f(x_1,\ x_2)=u+\Delta u.$$

The same process would show that by applying $(1+\Delta_{x_1})$ first, $(1+\Delta_{x_2})$ second, we would also get $u+\Delta u$. Hence would follow commutation of the two operators, so that

$$(1+\Delta_{x_1})(1+\Delta_{x_2})=(1+\Delta_{x_2})(1+\Delta_{x_1})=1+\Delta,$$

or if $u=f(x_1,\ x_2,\ \ldots\ldots\ x_n)$ we have proved that

$$(1+\Delta_{x_i})(1+\Delta_{x_k})=(1+\Delta_{x_k})(1+\Delta_{x_i})=1+\Delta_{x_i\,x_k},$$

i. e., any two of these operators are commutative. It follows $(n=2)$ that

$$u+\Delta_{x_2}u+\Delta_{x_1}u+\Delta_{x_1}\Delta_{x_2}u=u+\Delta_{x_1}u+\Delta_{x_2}u+\Delta_{x_2}\Delta_{x_1}u,$$

and since the ordinary addition is commutative, $\Delta_{x_1}\Delta_{x_2}u=\Delta_{x_2}\Delta_{x_1}u$, a familiar result.

Assume for $u=f(x_1,x_2\ldots\ldots x_n)$, that $u+\Delta u=(1+\Delta_{x_1}).\ \ldots.(1+\Delta_{x_n})u$.

Let $U=F(x_1,\ x_2\ldots\ldots x_{n+1})$.

Then by the assumption

$$U+\Delta_{x_1\ldots\ldots x_n}U=(1+\Delta_{x_1})(1+\Delta_{x_2}\ldots\ldots(1+\Delta_{x_n})U.$$

Apply $(1+\Delta_{x_{n+1}})$ to both sides of this equation remembering that we have proved commutation of operators. We get,

$$U+\Delta_{x_1\ldots\ldots x_n}U+\Delta_{x_{n+1}}U+\Delta_{x_{n+1}}\Delta_{x_1\ldots\ldots x_n}U=(1+\Delta_{x_1})\ldots\ldots(1+\Delta_{x_{n+1}})U.$$

By working out the left member, it becomes $U+\Delta U$, hence the next case takes the same form with respect to $n+1$, that the assumption had with respect to n, and since the assumption is true for $n=2$, it is true universally. The com-

mutation of any two operators brings with it the proof that $\Delta_{x_1}\Delta_{x_2}\ldots\ldots\Delta_{x_r}u$ is equal to any other one of $r!$ orders in which the r operations might be brought about. The determinant form was suggested by the formula for the development of a determinant in terms of the elements of its principal diagonal, a formula which has the same limits and number of operators in the summation. It is easily shown by adding to each column the elements of the last, when it reduces to one term, its principal diagonal term.

2. The second formula is easily shown after it has been proved for two variables. Let $u=f(x_1, x_2)$. Now u may be composed linearly of a constant, a function of x_1 alone, a function of x_2 alone, and a function of (x_1, x_2). The last must always be present, though the others may be wanting. About the constant we are not here concerned. Neglecting it,

$$u=f_1(x_1)+f_2(x_2)+\phi(x_1, x_2).$$

$$d_{x_1}u=[f'(x_1)+D_{x_1}\phi]d_{x_1}, \; d_{x_2}u=[f_2'(x_2)+D_{x_2}\phi]d_{x_2},$$

$$d_{x_1}d_{x_2}u=D_{x_1}D_{x_2}\phi dx_1dx_2. \quad \int d_{x_1}u=f_1(x_1)+\phi(x_1, x_2),$$

$$\int d_{x_2}u=f_2(x_2)+\phi(x_1, x_2). \quad \int\int d_{x_1}d_{x_2}u=\int\int D_{x_1}D_{x_2}\phi d_{x_1}d_{x_2}=\phi(x_1, x_2).$$

Hence without a constant,

$$u=\int d_{x_1}u+\int d_{x_2}u-\int\int d_{x_1}d_{x_2}u.$$

i. e., we have here complete indefinite integral of du, and

$$(1-\int d)=(1-\int d_{x_1})(1-\int d_{x_2})u=0,$$

where it is evident that there is a commutation of operators, since the ordinary additions are commutative, and $d_{x_1}d_{x_2}u=d_{x_2}d_{x_1}u$.

Assume for $u=f(x_1, x_2\ldots\ldots x_n)$, that

$$(1-\int d_{x_1})(1-\int d_{x_2})\ldots\ldots(1-\int d_{x_n})u=0. \quad \text{Let } U=F(x_1, x_2\ldots\ldots x_{n+1}).$$

Then $U=\int d_{x_1}U+\phi(x_2, x_3\ldots\ldots x_{n+1})$, where ϕ is an arbitrary function of all the variables but x_1. By the assumption

$$(1-\int d_{x_2})(1-\int d_{x_3})\ldots(1-\int d_{x_{n+1}})\phi=0, \text{ also } \phi=U-\int d_{x_1}U=(1-\int d_{x_1})U.$$

Apply the operator $(1-\int d_{x_2})\ldots\ldots(1-\int d_{x_{n+1}})$ to both sides of this equation, remembering that commutation of operators has been proved. We get

$$(1-\int d_{x_2})\ldots\ldots(1-\int d_{x_{n+1}})\phi=(1-\int d_{x_1})\ldots\ldots(1-\int d_{x_{n+1}})U=0,$$

since the left member is zero ; also the last has the same form with respect to $n+1$, that the assumption had with respect to n, and since the assumption is true when $n=2$, it is universally true.

III. Applications.

1. An elegant application of the first development is its use in demonstrating that the total differential of a function is equal to the sum of its partial differentials. Let $u=f(x_1, x_2 \ldots\ldots x_n)$. Then

$$\Delta u=\Delta_{x_1}u+\Delta_{x_2}u+\ldots\ldots\Delta_{x_n}u+\Sigma\Delta_{x_i}\Delta_{x_k}+\ldots\ldots\Sigma\Delta_{x_1}\Delta_{x_2}\ldots\ldots\Delta_{x_n}u.$$

Multiply through by n, where n is a number which becomes indefinitely great as the principal increment becomes indefinitely small, but so that $\lim_{\Delta x_r \doteq 0}\left(n\Delta_{x_r}u\right)=$a finite number, which is called by Hamilton, Serret, and J. M. Pierce, differential of u with respect to x_r, and may be denoted by $d_{x_r}u$. Consider $\lim_{\substack{\Delta x_r \doteq 0\\ \Delta x_k}}\left(n\Delta_{x_r}\Delta_{x_k}u\right)$. This equals $\lim_{\substack{\Delta x_r \doteq 0\\ \Delta x_k}}\left(\Delta_{x_r}n\Delta_{x_k}u\right)=\lim_{\Delta x_r \doteq 0}\Delta_{x_r}\left(d_{x_k}\right)=0.$

A *fortiori*, zero will be the limit of any term of higher order. We have then at once by taking the limits of both members,

$$du=d_{x_1}+d_{x_2}u+\ldots\ldots d_{x_n}u.$$

2. The second formula gives us the complete solution of a differential equation, when it is a perfect differential. The solution is,

$$u=\int d_{x_1}u+\int d_{x_2}u+\ldots\ldots\int d_{x_n}u-\sum\int\int d_{x_i}d_{x_k}u+\sum\int\int\int d_{x_i}d_{x_k}d_{x_j}u\ldots$$

$$+(-1)^{n-1}\int\int\ldots\ldots\int d_{x_1}d_{x_2}\ldots\ldots d_{x_n}u.$$

To this a constant may be added which is determined as usual by corresponding values of the variables and function. It may or may not be zero. When the function is such that farther differentiation of the first differentials will cut them down rapidly, this formula ought to be practically useful. When $n=2$, we can by this formula solve the problem of finding the orthogonal and isothermal curves to a given system, $u=c$, when u satisfies the equation $D_x{}^2u+D_y{}^2u=0$.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

73. Proposed by NELSON S. RORAY, South Jersey Institute, Bridgeton, New Jersey.

A man owes me \$100 due in 2 years, and I owe him \$200 due in 4 years. When can I pay him \$100 to settle the account equitably, money being worth 6%, and the interest to draw interest until the time of settlement?

Solution by FREDERIC R. HONEY, New Haven, Connecticut.

One dollar placed at 6% compound interest, in two years will amount to $1.06^2=\$1.1236$. Therefore the present value of \$100.00 due in 2 years is $\$100.00\div1.1236=\$89.00$ *very nearly.*

One dollar placed at 6% compound interest in four years will amount to $1.06^4=\$1.2625$. Therefore the present value of \$200.00 due in 4 years is $\$200.00\div1.2625=\$158.416$.

Therefore the difference between \$158.416 and $\$89.00=\$69.416$, is the amount of my debt to A at the present time.

Since \$1.00 placed at 6% compound interest in 6 years will amount to $1.06^6=\$1.4185$, \$69.416 at the same rate will, in 6 years, amount to $69.416\times 1.4185=\$98.4666$.

And since the simple interest on one dollar for 1 year is \$0.06, the simple interest on \$98.4666 is $98.4666\times0.06=\$5.908$ for one year. Therefore the interest $\$100.00-\$98.4666=\$1.5334$ will accrue in $1.5334\div5.908=0.2594$ years.

And $6.+0.2594=6.2594$ the number of years hence when \$100.00 should be paid, in order to settle the account equitably.

J. M. Bandy sent solutions of Nos. 71 and 72 too late for credit in February number.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

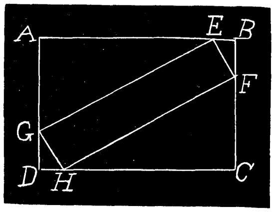
SOLUTIONS OF PROBLEMS.

67. Proposed by F. M. PRIEST, St. Louis, Mo.

Required: The length of a piece of carpet that is a yard wide with square ends, that can be placed diagonally in a room 40 feet long and 30 feet wide, the corners of the carpet just touching the walls of the room.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas ; P. S. BERG, Larimore, North Dakota ; J. SCHEFFER, A. M., Hagerstown, Maryland ; J. M. COLAW, A. M., Monterey, Virginia ; R. H. WAGONER, Westerville, Ohio ; J. F. YOTHERS, Westerville, Ohio ; J. T. FAIRCHILD, Crawfis College, Ohio ; CHAS. C. CROSS, Laytonsville, Maryland ; and O. S. WESTCOST, Chicago, Illinois.

Let $AB=40=a$, $BC=30=b$, $EF=3=c$, $BF=x$, $BE=y$.



$\therefore\ x^2+y^2=c^2$ ....(1).

From the triangles CHF and BEF we get $HC : CF=BF : BE$ or $a-y : b-x=x : y$.

$\therefore\ ay-y^2=bx-x^2$ ....(2).

(1) in (2) gives $bx-x^2=a\sqrt{c^2-x^2}-c^2+x^2$.

$\therefore\ 4x^4-4bx^3+(a^2+b^2-4c^2)x^2+2bc^2x+c^4-a^2c^2=0$.

$\therefore\ 4x^4-120x^3+2464x^2+540x-14319=0$.

$\therefore\ x=2.43372+$, $y=1.75414+$.

$HG=\{(a-y)^2+(b-x)^2\}^{\frac{1}{2}}=47.14494+$.

II. Solution by COOPER D. SCHMITT, Professor of Mathematics, University of Tennessee, Nashville, Tennessee ; W. H. HARVEY, Portland, Maine ; and B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

In the figure used above, let $AB=DC=a=40$ feet, the length of the room.

$AD=BC=b=30$ feet, the width of the room.

$EF=GH=c=3$ feet, the width of the carpet.

Let $x=HF=GE$, the length of the carpet, and the angle CFH=the angle $HGD=\theta$.

Then $x\sin\theta=CF$, $c\sin\theta=DH$, $x\cos\theta=HC$, and $c\cos\theta=DG$.

$\therefore\ c\sin\theta+x\cos\theta=DC=a$ ....(1),

and $x\sin\theta+c\cos\theta=BC=b$ ....(2).

Multiplying (1) by (2), and collecting, we get

$cx(\sin^2\theta+\cos^2\theta)+(x^2+c^2)\sin\theta\cos\theta=ab$, or

$cx+(x^2+c^2)\sin\theta\cos\theta=ab$ ....(3).

Squaring (1) and (2) and adding the results, we get

$c^2+x^2+4cx\sin\theta\cos\theta=a^2+b^2$ ....(4).

From (3), $\sin\theta\cos\theta=(ab-cx)/(x^2+c^2)$. Substituting this value of $\sin\theta\cos\theta$ in (4) and reducing, we get,

$$x^4-(a^2+b^2+2c^2)x^2+4abcx-c^2(a^2+b^2-c^2)=0 \qquad (5).$$

Restoring numbers in (5), we have

$$x^4-2518x^2+14400x-22419=0.$$

Solving this equation by Horner's Method, we find $x=47.145$ feet, nearly.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Virginia. All contributions to this department should be sent to him.

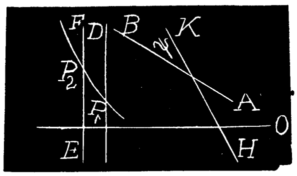
SOLUTIONS OF PROBLEMS.

58. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

A line passes through a fixed point and rotates uniformly about this point. Another line passes through a point which moves uniformly along the arc of a given curve and rotates uniformly about this point. Develop a method for finding the locus of intersection of these two lines. Apply to case of circle and straight line.

II. Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

Let O be the origin, P_3 the fixed point, its coördinates being (r_3, θ_3), and let AB be a given position of line through P_3. Let $P_1(r_1, \theta_1)$ be position of point on curve and CD the line through it, both corresponding to the position AB of other line. Also let HK be position of AB revolved through an $\angle\psi$, and let $P_2(r_2, \theta_2)$ and EF be the corresponding position of P_1 and CD.



Let $r=f(\theta)$ be equation to curve P_1P_2. Let η=the angle made by AB, and η_1 the one made by CD with a polar axis. Let a=angular rate of revolution of AB, and na of CD.

$\therefore \angle$ between CD and $EF=n\psi$.

Let b=linear rate of movement of P_1. Then $\psi/a=P_1P_2/b \qquad (1).$

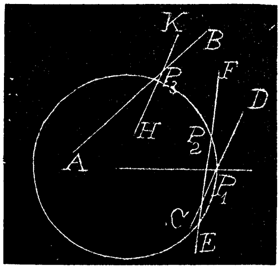
Equation to KH is $r=[r_3\sin(\eta+\psi-\theta_3)]/\sin(\eta+\psi-\theta) \qquad (2).$

Equation to EF is $r=(r_2\sin(\eta_1+n\psi-\theta_2)]/\sin(\eta_1+n\psi-\theta) \qquad (3).$

By integration,

$$P_1P_2=\int_{\theta_1}^{\theta_2}\sqrt{r^2+[(dr/d\theta)]^2}\,d\theta=\int_{\theta_1}^{\theta_2}\sqrt{[f(\theta)]^2+[f'(\theta)]^2}\,d\theta\ldots\ldots\ldots\ldots(4),$$

which gives P_1P_2 in terms of θ_2, θ_1 being known. Then substitute from (4) in (1) to get ψ in terms of θ_2. Substitute this value, and also $f(\theta_2)$ for r_2 in (2) and (3). Then by eliminating (θ_2) we have resulting the equation to the locus of the intersecting of the lines. The solution depends on our ability to integrate (4). Now if the given lines are not straight, it is evident that the only changes are in equations (2) and (3). These may be derived from the equations to the lines in original position by a method of transformation of coördinates. For example, the equation to HK may be derived from that to AB by revolving the pole and polar axis about P_3 through an angle equal to ψ and in the opposite direction. If the given curve is a circle and the lines straight, the problem can be definitely solved as follows:



Transform coördinates so that center of circle shall be pole and OP_1 the the polar axis. Then $r=f(\theta)$ becomes $r=c$.

Let $(r_1,\ \theta_1)(r_2,\ \theta_2)(r_3,\ \theta_3)$ represent the new coördinates of points P_1, P_2, P_3, respectively.

Then $r_1=r_2=c$. $\theta_1=0$. $P_1P_2=c\theta_2$. From (1) $\psi=ac\theta_2/b$.

(2) becomes $r=[r_3\sin(\eta+ac\theta_2/b-\theta_3)]/\sin(\eta+ac\theta_2/b-\theta)$. . . . . (5).

(3) becomes $r=[c\sin(\eta_1+nac\theta_2/b-\theta_2)]/\sin(\eta_1+nac\theta_2/b-\theta)$. . . . . (6).

(5) can be solved for θ_2 and the result can be substituted in (6), giving the equation required. Then if desired the coördinates can be again transformed to the original form.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

43. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

Two weights P and Q rest on the concave side of a parabola whose axis is horizontal, and are connected by a string, length l, which passes over a smooth peg at the focus, F. [*Bowser's Analytical Mechanics*, page 54.]

I. Solution by **ALFRED HUME**, C. E., D. Sc., Professor of Mathematics, University of Mississippi; **COLMAN BANCROFT**, M. Sc., Professor of Mathematics, Hiram College, Hiram, Ohio; **GEORGE LILLEY**, LL. D., Portland, Oregon; **G. B. M. ZERR**, A. M., Ph. D., Texarkana, Arkansas, and the **PROPOSER**.

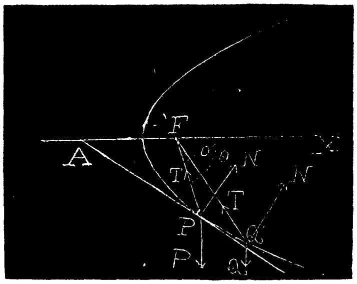
Let AX be the horizontal axis of the parabola, F the focus, P' and Q' the positions of the weights P and Q, T the tention of the string $P'FQ'$, N and N' the normal reactions at P' and Q' respectively, θ and θ' the angles between the axis AX and the focal radii to P' and Q' respectively, A the intersection of the axis and the tangent at P'. Denote the latus rectum by $4m$.

$$\angle\theta=\angle FAP'+\angle FP'A=2\angle FAP',$$

by property of parabola.

$\therefore\ \angle FAP'=\frac{1}{2}\theta.$

Since N is inclined to the vertical at the same angle that the tangent is inclined to the horizontal, we have for equilibrium of the forces at P', resolving vertically and horizontally,



$N\cos\frac{1}{2}\theta+T\sin\theta=P$, and $N\sin\frac{1}{2}\theta=T\cos\theta$,

from which

$T(\cot\frac{1}{2}\theta\cos\theta+\sin\theta)=P$, or, $T\cot\frac{1}{2}\theta=P$.

Similarly, $T\cot\frac{1}{2}\theta'=Q$. $\therefore\ \cot\frac{1}{2}\theta=P/Q\cot\frac{1}{2}\theta'$.

From the polar equation of a parabola,

$$FP'=\frac{2m}{1-\cos\theta},\quad FQ'=\frac{2m}{1-\cos\theta'}.$$

But $FP'+FQ'=l$. $\therefore\ \dfrac{2m}{1-\cos\theta'}=l-\dfrac{2m}{1-\cos\theta'}$, $\cos\theta'=1-\dfrac{2m(1-\cos\theta)}{l(1-\cos\theta)-2m}$,

$$\sin^2\tfrac{1}{2}\theta'=\frac{m(1-\cos\theta)}{l(1-\cos\theta)-2m},\quad \cot\tfrac{1}{2}\theta'=\sqrt{\frac{l(1-\cos\theta)+m\cos\theta-3m}{m(1-\cos\theta)}}.$$

Then, $\cot\frac{1}{2}\theta=\dfrac{P}{Q}\sqrt{\dfrac{l(1-\cos\theta)+m\cos\theta-3m}{m(1-\cos\theta)}}$. Since $1-\cos\theta=\dfrac{2}{\cot^2\frac{1}{2}\theta+1}$, the preceding equation gives $\cot\frac{1}{2}\theta=\dfrac{P\sqrt{l-2m}}{\sqrt{m(P^2+Q^2)}}$.

II. Solution by **F. ANDEREGG**, A. M., Professor of Mathematics, Oberlin College, Oberlin, Ohio.

If, in the figure above, θ represents the angle XFP, and θ_1, XFQ, then

$$FP=r=\frac{2m}{1-\cos\theta}=\frac{m}{\sin^2\frac{1}{2}\theta}.\quad\therefore\ \sin^2\tfrac{1}{2}\theta=\frac{m}{r},\ \text{and}\ \cos^2\tfrac{1}{2}\theta=\frac{r-m}{r}.$$

Since $FQ=l-r$, $\sin^2\frac{1}{2}\theta_1=\dfrac{m}{l-r}$, and $\cos^2\frac{1}{2}\theta_1=\dfrac{l-r-m}{l-r}$.

Since the tension is the same in all parts of the string and the angle between the radius vector and tangent is half the angle between the radius vector and the X axis, $T=P\tan\frac{1}{2}\theta=Q\tan\frac{1}{2}\theta_1$.

$$\therefore \frac{P}{Q}=\frac{\text{ctn}\frac{1}{2}\theta}{\text{ctn}\frac{1}{2}\theta_1},\quad \therefore \frac{P^2}{P^2+Q^2}=\frac{r-m}{l-2m}=\frac{m\,\text{ctn}^2\frac{1}{2}\theta}{l-2m},$$

$$\therefore\ \text{ctn}\tfrac{1}{2}\theta=\frac{P\sqrt{l-2m}}{\sqrt{m(P^2+Q^2)}}.$$

III. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, Ohio.

Let r', r'' be the parts of the string l joining the focus and the weights P and Q; θ and θ' the angles which r' and r'' make with the axis of X.

For the equilibrium of P and Q, resolving along the tangents through P and Q, T being the tension in the string,

$T\sin\frac{1}{2}\theta=P\cos\frac{1}{2}\theta$................(1). $T\sin\frac{1}{2}\theta'=Q\cos\frac{1}{2}\theta'$..... .... (2).

These give $P/Q\cot\frac{1}{2}\theta=\cot\frac{1}{2}\theta'$........... ....(3).

The equations to the curve are

$$r'=\frac{2m}{1+\cos\theta},\quad r''=\frac{2m}{1+\cos\theta'};$$

then $r'+r''=2m\left(\dfrac{1}{1+\cos\theta}+\dfrac{1}{1+\cos\theta'}\right)=l$..... .... (4).

Now $\cos\theta=\dfrac{\cot^2\frac{1}{2}\theta-1}{\cot^2\frac{1}{2}\theta+1}$, $\cos\theta'=\dfrac{\cot^2\frac{1}{2}\theta'-1}{\cot^2\frac{1}{2}\theta'+1}$ ....(5).

(3) and (5) and the resulting values of $\cos\theta$ and $\cos\theta'$ in (4) and reducing gives

$$\frac{2P^2\cot^2\frac{1}{2}\theta+(P^2+Q^2)}{2P^2\cot^2\frac{1}{2}\theta}=\frac{l}{2m}\ \ldots\ldots\ldots\ldots(6).$$

Subtracting unity from both members of (6) and taking the square root of the result,

$$\tan\tfrac{1}{2}\theta=\frac{P}{\sqrt{P^2+Q^2}}\sqrt{\frac{l-2m}{m}}\ \ldots\ldots\ldots\ldots(7).$$

In which put $\pi-\theta$ for θ for Bowser's result.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

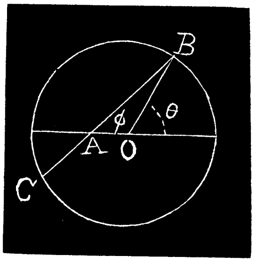
SOLUTIONS OF PROBLEMS.

43. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

In a circle whose radius is a, chords are drawn through a point distant b from the center. What is the average length of such chords, (1), if a chord is drawn from every point of the circumference, and (2), if they are drawn through the point at equal angular intervals?

Solution by the PROPOSER.

In the figure, let BC represent the chord passing through the point A whose distance from O is $OA=b$. Put $BC=x$, $\angle BOA=\theta$, $\angle BAO=\phi$, A_1=first average required, and A_2=second average required.



Then $x=2(a^2-b^2\sin^2\phi)^{\frac{1}{2}}$.

Hence, $A_1=1/\pi\int_0^\pi xd\theta$. From triangle AOB,

$a\sin(\theta+\phi)=b\sin\phi$. $\therefore$ $\theta+\phi=\sin^{-1}(b/a\sin\phi)$.

$$\therefore\ d\theta=-d\phi+\frac{b\cos\phi d\phi}{(a^2-b^2\sin^2\phi)}.$$

$$\therefore\ A_1=2/\pi\int_0^\pi[(a^2-b^2\sin^2\phi)^{\frac{1}{2}}-b\cos\phi]d\phi=\frac{4a}{\pi}\ E\left(\frac{b}{a},\ \tfrac{1}{2}\pi\right).$$

$$A_2=1/\pi\int_0^\pi xd\phi=\frac{4a}{\pi}\int_0^{\frac{1}{2}\pi}[1-(b^2/a^2)\sin^2\phi]^{\frac{1}{2}}\,d\phi=\frac{4a}{\pi}\ E\left(\frac{b}{a},\ \tfrac{1}{2}\pi\right).$$

$\therefore$ $A_1=A_2$. If $b=0$, $A_1=A_2=2a$. If $b=a$, $A_1=A_2=4a/\pi$.

This problem was also solved by G. B. M. Zerr. His solution will appear in the next issue.

44. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

What is the average length of all the chords that may be drawn from one extremity of the major axis of an ellipse if they are drawn at equal angular intervals?

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas; J. SCHEFFER, A. M., Hagerstown, Maryland, and the PROPOSER.

Let r=length, e=eccentricity of ellipse, θ=angle the chord makes with major axis. Then

$$r=\frac{2a(1-e^2)\cos\theta}{1-e^2\cos^2\theta}.\quad \varDelta=\text{average length}=\frac{\int_0^{\frac{1}{2}\pi}rd\theta}{\int_0^{\frac{1}{2}\pi}d\theta}.$$

$$\therefore \Delta = \frac{4a(1-e^2)}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\cos\theta d\theta}{1-e^2\cos^2\theta}. \quad \therefore \Delta \frac{4a\sqrt{1-e^2}}{\pi e}\tan^{-1}\frac{e}{\sqrt{1-e^2}}.$$

When $e=\theta$, $\Delta=\dfrac{4a}{\pi}$, since $\dfrac{1}{e}\tan^{-1}\dfrac{e}{\sqrt{1-e^2}}=1$.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Virginia. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

42. Proposed by E. B. ESCOTT, 6123 Ellis Avenue, Chicago, Illinois.

To find a triangle whose sides and median lines are commensurable.

Solution by J. W. TESCH, in "L'Intermediare des Mathematiciens" for October, 1896. Translated and adapted by J. M. COLAW, A. M., Monterey, Virginia.

Suppose the sides to be $2a$, $1+2b-b^2+\frac{1}{4}a^2$, $1-2b-b^2+\frac{1}{4}a^2$; then we will have $m_1=\pm(1+b^2-\frac{1}{4}a^2)$,

$m_2{}^2=\frac{1}{2}[4a^2+(1-2b-b^2+\frac{1}{4}a^2)^2]-\frac{1}{4}(1+2b-b^2+\frac{1}{4}a^2)^2$,

or $m_2{}^2=\frac{1}{64}a^4+\frac{1}{8}(17-6b-b^2)a^2+\frac{1}{4}(1-12b+2b^2+12b^3+b^4)$.

In order that the second member may be a perfect square, it is necessary that $\frac{1}{64}(17-6b-b^2)^2=4\times\frac{1}{64}\times\frac{1}{4}(1-12b+2b^2+12b^3+b^4)$, whence $2b=3$.

Thus the sides become $2a$, $\frac{7}{4}+\frac{1}{4}a^2$, $-\frac{17}{4}+\frac{1}{4}a^2$, or

(1) $16a$, $2(a^2+7)$, $\pm2(a^2-17)$; $m_1=\pm2(a^2-13)$, $m_2=a^2+23$.

We will have $m_3{}^2=a^4+190a^2-191$. The values of a, which render the second number a perfect square, are 1, 3, 5, ... ; $m_3=0$, 40, 72, ... ; but none of these values satisfy (1); therefore, after the method of Euler, (*Vollständige Anleitung zur Algebra*, or the French translation by J.-G. Garnier, 2 vols., with the additions of Lagrange, Paris, 1807), it is necessary to proceed as follows:

By supposing $a=3+h$, we may write

$$(3+h)^4+190(3+h)^2-191=(40+ph+h^2)^2,$$

where p is a coefficient to be determined. Developing, we have

$$1248+244h+12h^2=80p+(80+p^2)h+2ph^2.$$

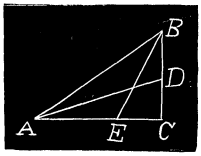
If we take $80p=1248$ or $p=15+3/5$, we find $h=-(4+2/15)$, which gives $a^2=(17/15)^2$. We also have for the three sides, after some easy reductions: 510, 466, 884, and for the medians 659, 683, 208. This is perhaps the simplest case in whole numbers.

43. **Proposed by H. C. WILKES, Skull Run, West Virginia.**

To find, if possible, a right angled triangle, the bisectors of the acute angles of which, can be expressed by integral whole numbers.

I. **Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.**

Let ABC be a right triangle, right angled at C, AD the bisector of $\angle A$, and BE the bisector of $\angle B$.



Put $BC=a$, $AC=b$, $AB=c$, $DC=a_1$, $EC=b_1$, $EB=c_1$, and $AD=c_2$. Then $BD=a-a_1$, and $AE=b-b_1$. From geometrical relations we obtain $a^2+b^2=c^2$ ......(1); $c_1{}^2=ac-b_1(b-b_1)$......(2); $b-b_1 : b_1=c : a$......(3).

From (3) we get $b : b_1=c+a : a$; whence $b_1=ab/(c+a)$, and $b-b_1=bc/(c+a)$.

$\therefore$ $c_1{}^2=ac-ab^2c/(c+a)^2=ac-ac(c^2-a^2)/(c+a)^2=2a^2c/(c+a)$.

By a similar process, we find $c_2{}^2=2b^2c/(c+b)$.

From (1), $c^2-b^2=a^2$, or $(c+b)(c-b)=a^2$. Put $c+b=tp^2$ and $c-b=tq^2$, t, p, and q being any values. Then $a=tpq$, $b=t(p^2-q^2)/2$, and $c=t(p^2+q^2)/2$. Whence $c_1{}^2=2t^2p^2q^2(p^2+q^2)/(p+q)^2$, and $c_2{}^2=t^2(p^2-q^2)^2(p^2+q^2)/4p^2$.

When $p^2+q^2=\square$, $c_2{}^2=\square$, and $c_1{}^2=2\times\square$. When $p^2+q^2=2\times\square$, $c_2{}^2=2\times\square$, and $c_1{}^2=\square$.

$\therefore$ *Both* bisectors cannot be rational; *one* of them will be $\sqrt{2}$ times a number, when the *other* is a rational whole number.

II. **Solution by the PROPOSER.**

Let bx, by, and $x+y$ be, respectively, the sides and base of a right angled triangle, and let x and y be the greater and less segments of the base cut by the bisector. Then the bisector will be $\sqrt{y^2(b^2+1)}$ and if the bisector be integral, b^2+1 must$=\square$. b must therefore be an improper fraction, and will always be the quotient of the sum of the other two sides divided by the bisected side.

Now let CAB be a triangle, and let $AB=x^2+y^2$, $CA=x^2-y^2$ and $CB=2xy$, $CA+AB/CB=x/y$. $(x/y)^2+1$ may be a square, but $AB+CB/CA=(x+y)/(x-y)$. $[(x+y)/(x-y)]^2+1$ will be a multiple of the $\sqrt{2}$ and cannot be a square.

$\therefore$ If a rational right angled triangle have an integral bisector of one of its acute angles, the bisector of the other acute angle must be a multiple of $\sqrt{2}$ and and cannot be integral.

[Remark.—On page 155, Vol. II. of the MONTHLY, we have, when the sides are 59.4107, 47.4072, 35.8067, the bisectors 40 and 50. It is doubtful whether the sides and bisectors both can be integral. ZERR.]

44. Proposed by **P. S. BERG**, Principal of Schools, Larimore, North Dakota.

Two trees whose heights are 40 and 80 feet, respectively, stand on opposite sides of a stream 30 feet wide. What path does a squirrel take in leaping from the top of the higher to the top of the lower? What is the length of the path?

Solution by **G. B. M. ZERR, A. M., Ph. D.**, Texarkana, Arkansas.

The path is a parabola. Let the top of the higher tree be origin, "across the river" positive, v=velocity, β=angle of projection, then the equation is $y=\tan\beta x-gx^2/(2v^2\cos^2\beta)$, in which we must know either v, or β. Substituting $x=30$, $y=-40$ we get

$$v^2=45g/(4\cos^2\beta+3\sin\beta\cos\beta)\ldots\ldots\ldots\ldots(1).$$

$$S=\frac{1}{v^2\cos^2\beta}\int_0^{30}\sqrt{v^4\cos^4\beta+(gx-v^2\sin\beta\cos\beta)^2}\,dx.$$

Let $v^2\cos^2\beta=a$, $gx-v^2\sin\beta\cos\beta=y$, $30g-v^2\sin\beta\cos\beta=y_1$, $-v^2\sin\beta\cos\beta=-y_2$.

$$\therefore S=\frac{1}{ag}\int_{-y_2}^{y_1}\sqrt{a^2+y^2}\,dy$$

$$=\frac{1}{2ag}\{y_1\sqrt{a^2+y_1^{\,2}}+y_2\sqrt{a^2+y_2^{\,2}}\}+\frac{a}{2g}\log[(\sqrt{a^2+y_1^{\,2}}+y_1)/(\sqrt{a^2+y_2^{\,2}}-y_2)]$$

Let $\beta=45^\circ$; then $v^2=90g/7$, $y_1=165g/7$, $y_2=45g/7=a$.

$\therefore S=\frac{5}{14}(11\sqrt{130}+9\sqrt{2})+\frac{45}{14}\log[(\sqrt{130}+11)/(3\sqrt{2}-3)]$.

Let $\beta=0$; then $v^2=45g/4$, $y_1=30g$, $y_2=0$, $a=45g/4$.

$\therefore S=5\sqrt{73}+\frac{45}{8}\log[(\sqrt{73}+8)/3]$.

NOTES.

INTERNATIONAL CONGRESS OF MATHEMATICIANS AT ZURICH IN 1897.

"It is known that the idea of an international congress of mathematicians has been, above all in these latter days, the object of numerous deliberations on the part of scientists interested in its realization. It has appeared to them, by reason of the excellent results obtained in other scientific domains by an international 'entente', that assuring the execution of this project would have very weighty advantages.

As outcome of a very active exchange of views, accord was reached on a

prime point. Switzerland, by its central geographic situation, by its traditions and its experience of international congresses, appeared designated to invite a first attempt at a reunion of mathematicians. In consequence Zurich is chosen as seat of the Congress.

The mathematicians of Zurich do not disguise from themselves the difficulties they will have to surmount. But in the interest of this enterprise, they have thought it their duty not to decline the overtures so flattering that have been made them from all sides. They have decided therefore to take all preparatory measures for the future congress and, to the extent of their powers, to contribute to its success. So, with the concurrence of mathematicians of other nations, was formed the undersigned committee of organization, charged *to bring together at Zurich in* 1897 *the mathematicians of the entire world.*

The congress, in which you are cordially invited to take part, will take place at Zurich the 9, 10 and 11 of August, 1897, in the halls of the federal polytechnic school. The committee will not fail to communicate to you, in time opportune, the text of the program determined, begging you to inform them of your adherence. But even at present it may be said that the scientific works and questions of administration will pertain to subjects of general interest or recognized importance.

Scientific congresses have also this precious advantage, to favor and keep up personal relations. The local committee will not fail to accord all its solicitude to this part of its task, and, with this aim, it will elaborate a modest programme of fètes and intimate reunions.

May the hopes reposed in this first congress be fully realized! May numerous participants contribute by their presence to create, among colleagues, not alone coherent scientific relations, but also cordial bonds based on personal acquaintance! Finally, may our congress serve the advancement and the progress of the mathematical sciences!"

The invitation of which the above is a translation is signed by eleven from Zurich and ten associates, as committee.

Readers of the AMERICAN MATHEMATICAL MONTHLY already know the persistent efforts of Vasiliev of Kazan and Laisant of Paris to establish this congress.

It is matter for rejoicing that their noble endeavors have been crowned with this definite success. GEORGE BRUCE HALSTED.

THE SAME OLD BLUNDER.

In the *Nation* of November 26, 1896, in a review of Cajori's History of Elementary Mathematics, the reviewer himself makes a blunder so appalling that it should not go unnoticed.

He says Cajori "does not name Prof. J. J. Littrow of Vienna, whose demonstration is yet worth notice. Littrow proves first that the three angles of a triangle are $=2R$. Thus: When a side a and angles BC are given, angle A is determined; it is $=F(aBC)$; and as an angle may be viewed as an abstract number, it has no relation to one measure in space: angle $A=F(BC)$ simply," etc.

Now where has the *Nation's* reviewer been buried not to know that this very pseudo-proof was given by Legendre, and its fallacy shown by George Paxton Young in the *Canadian Journal* for November, 1856, forty years ago, and again in the *Canadian Journal* for July, 1860, pages 356-358 ?

GEORGE BRUCE HALSTED.

Austin, Texas.

James Joseph Sylvester, the great mathematician, Sivilian professor of geometry at Oxford, formerly professor of mathematics at the Johns Hopkins University, and in 1841 at the University of Virginia, died in London on March 15th, aged eighty-three years. Also the eminent mathematician Dr. Karl Weierstrass, died at Berlin on February 19th, aged eighty-one years. In the death of these two men, mathematics sustains a great loss. Both did much to broaden and deepen mathematical knowledge. Sylvester has written much on invariants, the theory of equations, theory of partitions, multiple algebra, the theory of numbers, the theory of reciprocants, etc., while Weierstrass has given special attention to the theory of functions of a complex variable. For a biography of Sylvester, by Dr. Halsted, see MONTHLY, Vol. I., No. 9. B. F. F.

BOOKS.

Composite Geometrical Figures. By George A. Andrews, A. M. 63 pages. Price 55 cents. Ginn and Company, Boston, and London. 1896.

The figures in this little book are constructed for the demonstration of more than one proposition. Ten of the figures are designed for review work, while the last general figure is intended for re-review work of all the theorems of plane geometry. Under each figure are illustrative demonstrations, followed by series of easy examples which require the pupil to apply the general principles of geometry to the specific conditions of the figures. It will be seen that the book is not designed to take the place of other text-books, but is intended to be used with them for reviews and for supplemental easy original work. The plan of the work is rather unique, and it will be useful to teachers who feel the need in their classes for the specific application of geometrical principles. J. M. C.

National Geographic Monographs: (1) *Physiographic Processes*, by J. W. Powell ; (2) *Physiographic Regions of the U. S.*, by J. W. Powell ; (3) *Lakes and Sinks of Nevada*, by I. C. Russell; (4) *Mt. Shasta*, by J. S. Diller. Price 20 cents each, or $1.50 for a set of ten. American Book Company, New York, Cincinnati, and Chicago.

These monographs on the physical features of the earth's surface furnish fresh and interesting material with which to supplement the regular text books. They have been written with exceptional care and ability and are not only very serviceable for such use, but are very interesting to the general reader as well. J. M. C.

THE

AMERICAN

MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. IV. APRIL, 1897. No. 4.

BIOGRAPHY.

HOÜEL.

BY GEORGE BRUCE HALSTED.

GUILLAUME JULES HOÜEL, of a very old protestant family of Normandy, was born at Thaon (Calvados) on April 7th, 1823, and died at Périers, near Caen, June 14th, 1886.

The key to his whole mental life was this old protestant blood, which means so much in a Roman catholic country.

After studying at the lyceum of Caen, and the college Rollin, he entered the great Normal School of Paris in 1843. On leaving, he taught at Bourges, Bordeaux, Pau, Alençon and Caen.

In 1855 he took his Doctor's degree at the Sorbonne, and then, declining the overtures of Le Verrier to join the working force at the Observatory, he retired to his home at Thaon to continue his researches. In 1859 he was called to succeed Le Besgue in the chair of pure mathematics of the Faculty of Sciences of Bordeaux. Here he found dignity and facilities for work, and considered the position as final.

The idea of duty was the essence of his character, remarkably sweet and even. Not only in his official position did he forward science, he spread it with profusion all around him.

So precise and rigorous was his mind that he scorned Legendre's and Clairant's geometries, and the conventional neatness of the French texts, in favor of the eternal geometer Euclid.

In 1863 he published at Greifswald an essay on the fundamental principles of geometry. In this he has already reached by himself the idea that a demonstration of the postulatum of Euclid is impossible. He says: "Since long, the scientific researches of mathematicians on the fundamental principles of elementary geometry have concentrated themselves almost exclusively on the theory of parallels; and if, hitherto, the efforts of so many eminent minds have produced no satisfactory result, it is perhaps permitted to conclude thence that in pursuing these researches they have followed a false path and attacked an insoluble problem, of which the importance has been exaggerated in consequence of inexact ideas on the nature and origin of the primordial verities of the science of space."

To the mind so self-prepared came an important communication in 1866 from Dr. R. Baltzer informing Hoüel of the fundamental idea of Lobachévski and Bolyai and announcing that Baltzer would mention it in the forthcoming second edition of his Elements of Geometry. That very year 1866 Hoüel issued his translation of Lobachévski's "Geometrische Untersuchungen zur Theorie der Parallellinien," and in the preface to his translation quotes from W. Bolyai's "Kurzer Grundriss eines Versuchs etc.," and mentions the work of J. Bolyai with date 1832. In this preface he says: "The aim of the author is to prove that there exists *à priori* no reason to affirm that the sum of the three angles of a rectilineal triangle is not less than two right angles, or, what comes to the same thing, that one cannot draw, through a given point more than a single straight not meeting a given straight in the same plane.

In spite of the high value of these researches, they have not hitherto drawn the attention of any geometer. We do not believe however that we exaggerate their philosophic import in saying that they throw a new day on the fundamental principles of geometry, and that they open a path yet unexplored capable of leading to unexpected discoveries. Not to go beyond elementary questions, one cannot deny that they accomplish an immense advance in methods of teaching by relegating among the chimeras the hope still nourished by so many geometers of demonstrating the postulatum of Euclid.

Henceforth these attempts must be ranked with the quadrature of the circle and perpetual motion."

He mentions the assumption, (three points are costraight or concyclic), given by W. Bolyai to replace Euclid's·

A translation of J. Bolyai was delayed until 1868 by Hoüel's inability to procure a copy of the now celebrated Appendix. How this difficulty was fortunately overcome I learned while in Hungary where my friend Franz Schmidt entrusted to me a precious file of Hoüel's own letters. From these letters it appears that a copy of Hoüel's 'Essai' of 1863 having come by chance into the hands of a young architect of Temesvár in Hungary, this youth (Franz Schmidt), desirous of continuing his mathematical studies wrote for counsel to Hoüel. Hoüel had answered helpfully, and later implored the aid of Schmidt to procure Bolyai's work, and besought Schmidt to collect what materials he could for a biography. This Schmidt did, and his article on Grunert's Archiv, 1868,

remained, until my own researches and my journey to Hungary, the only source of information on these wonderful Magyars. Schmidt succeeded in procuring for Hoüel two copies of Bolyai's work. One Hoüel proceeded to translate himself; the other he sent to Battaglini, asking him to make known in Italy this wonderful idea. This he did by an Italian translation. Thus to Hoüel belongs a perfectly definite and permanent place in the final history of human thought.

Much else he did; so much that I could not attempt to enumerate it in the brief space at my disposal here. Fortunately it has been most sympathetically done by M. G. Brunel in a book of 78 pages most obligingly furnished me by Hoüel's son-in-law, Monsieur H. Barckhausen.

M. Brunel cites on page 34 my Bibliography of Hyper-Space and Non-Euclidean Geometry (1878), and also that published at Kiev in 1880 by Vashtchenko-Zaharchenko, but omits to state that this latter was simply a reprint of mine with slight additions, as is also that given at the end of the Kazan edition of Lobachévski's Works, 1886. Some grotesque effects are produced by reprinting or attempting to reprint my English. Thus under P. G. Tait the title of the work is given as follows: "Mentions Hyper-Space in his Address as Pres. of Math. Sect. of Brit. Assoc. at Edinbvrgh." Under G. P. Young we read "The relation which can be proved to subsist between the Area of a Plane Triangle and the Sum of the Hypothesis that Euclid's twelfth Axiom is false." I must take it, from this extraordinary summation of a hypothesis, that English is nearly as difficult as Russian, though neither can for one instant compete with the Magyar.

In his personal character Hoüel reached that perfection which he has done so much to introduce into the foundations of Geometry.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M. (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from March Number.]

PROPOSITION XXVI. *If the aforesaid AX, BX (fig. 31.) must indeed meet each other, but only at their infinite production toward the parts of the point X: I say there will be no assignable point T in AB, from which a perpendicular erected towards the parts of AX does not at a finite or terminated distance meet this AX in some point F.*

Demonstratur. For (from the preceding hypothesis) there will be in AX

some point N, from which the perpendicular NK let fall to BX is less than any assigned length, as suppose this TB.

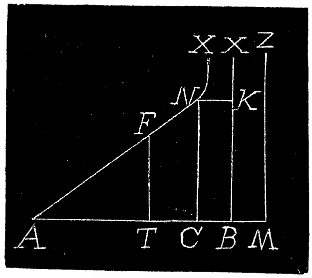


Fig. 31.

But then is assumed in TB a portion CB equal to NK, and CN is joined. In the hypothesis of acute angle it is known that the angle NCB will be acute. Therefore (from Eu. I. 13) NCT, which is the adjacent angle, will be obtuse.

Therefore the straight which is erected toward the parts of AX perpendicularly from the point T (disposed between the points A and C), does not meet (from Eu. I. 17) CN at any point; and therefore (lest it should enclose a space with AT; or with TC) it strikes the terminated AN in some point F.

Therefore even in the hypothesis of acute angle (which we know can here alone hinder) there will be in this AB no assignable point T, from which the perpendicular erected toward the parts of AX does not, at a finite or terminated distance, meet this AX in a certain point F. Quod etc.

Corollary I. But thence follows, that, point M being assumed in AB produced, from which towards the parts of the point X is erected a perpendicular MZ, this cannot, even if infinitely produced, meet the aforesaid AX; because otherwise that other straight BX must (from the foregoing demonstration) at a finite distance meet this AX; which is against the present hypothesis.

Corollary II. From which again follows, that every perpendicular erected from any point, but not however infinitely removed, of this AB produced indefinitely, must at a finite distance meet the aforesaid AX, as soon as indeed it is assumed that *every* such perpendicular ever more, without any certain limit, approaches the other ever produced straight AX.

Corollary III. Whence finally follows, that not even at its infinite production can BX be cut by that AX; because otherwise from any point of that AX beyond the aforesaid intersection a certain perpendicular ZM could be supposed let fall to AB produced; whence again would follow, that BX (against the present hypothesis) met the aforesaid AX not at an infinite, but wholly at a finite distance.

But this last dictum is beyond necessity.

[Saccheri here handles a point at infinity, or *figurative* point, as if it were a *proper* point. Upon the extent to which he realized this to be unallowable, depends his real mental attitude toward the non-Euclidean geometries he had discovered. Did he intend his work to suggest what he would not have been allowed to print ?]

A METHOD FOR DEVELOPING $\cos^n\theta$ AND $\sin^n\theta$.

By M. C. STEVENS, A. M., Department of Mathematics, Purdue University, Lafayette, Indiana.

De Morgan in his Calculus gives a method for expanding $\cos^n\theta$ and $\sin^n\theta$ when n is an integer which I have not noticed in any of our American works on that subject. As it leads to an easy method for integrating such expressions as

$$\int\cos^n\theta d\theta,\quad \int\sin^n\theta d\theta,$$

etc. I have thought it might be of interest to some of the readers of the MONTHLY.

The method is as follows:

$$\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}.\quad \text{Let } e^{i\theta}=x, \text{ then } e^{-i\theta}=\frac{1}{x}, \text{ and } \cos\theta=\tfrac{1}{2}\left(x+\frac{1}{x}\right)\ldots\ldots\ldots\ldots(1)$$

$$e^{ni\theta}=x^n, \text{ then } e^{-ni\theta}=\frac{1}{x^n},\ \cos n\theta=\tfrac{1}{2}\left(x^n+\frac{1}{x^n}\right).$$

$$\text{Then from (1) } \cos^n\theta=\frac{1}{2^{n-1}}\left[\tfrac{1}{2}\left(x^n+\frac{1}{x^n}\right)+n\tfrac{1}{2}\left(x^{n-2}+\frac{1}{x^{n-2}}\right)\right.$$

$$\left.+\frac{n(n-1)}{2}\tfrac{1}{2}\left(x^{n-4}+\frac{1}{x^{n-4}}\right)\ldots\ldots\right.$$

$$=\frac{1}{2^{n-1}}\left[\cos n\theta+n\cos(n-2)\theta+\frac{n(n-1)}{2}\cos(n-4)\theta+\ \ldots\ldots\right.$$

If n be an even number$=2m$, there will be $2m+1$ terms in the development, which will give m cosines, namely, those of $2m\theta$, $2(m-1)\theta\ldots\ldots$down to 2θ, and an additional term which will not contain θ, the value of which is $\frac{2m(2m-1)\ldots\ldots m+1}{\underline{|m}}$. But if n be odd, and$=2m+1$, then there are $2m+2$ terms giving $m+1$ cosines, namely, those of $(2m+1)\theta$, $(2m-1)\theta\ldots\ldots\ldots$ down to θ, with no middle term. Thus we have

$$\cos^6\theta=\tfrac{1}{32}(\cos6\theta+6\cos4\theta+15\cos2\theta+10).$$

Whence the integral of $\cos^6\theta d\theta=\frac{1}{192}\sin6\theta+\frac{3}{64}\sin4\theta+\frac{15}{64}\sin2\theta+\frac{5}{16}\theta$.

Also $\cos^7\theta=\frac{1}{64}(\cos7\theta+7\cos5\theta+21\cos3\theta+35\cos\theta)$.

Whence $\int\cos^7\theta d\theta=\frac{1}{448}\sin7\theta+\frac{7}{320}\sin5\theta+\frac{7}{64}\sin3\theta+\frac{35}{64}\sin\theta$.

The advantage of this method will be still more apparent by integrating* $\cos^3 3\theta \cos\theta d\theta$. Here $\cos^3 3\theta = \frac{1}{8}(x^3 + x^{-3})^3 = \frac{1}{8}(x^9 + x^{-9}) + 3(x^3 + x^{-3})$.

Multiplying this by $\frac{1}{2}(x + x^{-1})$ we at once have

$$\cos^3 3\theta \cos\theta = \tfrac{1}{8}\cos 10\theta + \tfrac{1}{8}\cos 8\theta + \tfrac{3}{8}\cos 4\theta + \tfrac{3}{8}\cos 2\theta.$$

Whence $\int \cos^3 3\theta \cos\theta d\theta = \frac{1}{80}\sin 10\theta + \frac{1}{64}\sin 8\theta + \frac{3}{32}\sin 4\theta + \frac{3}{16}\sin 2\theta$.

It will be noticed that this form is well adapted for substituting values as limits of integration. For instance if the inferior limit be 0, and the superior limit $\frac{1}{6}\pi$ then $\frac{1}{80}\sin\frac{10}{6}\pi = -\frac{1}{160}\sqrt{3}$; $\frac{1}{64}\sin\frac{8}{6}\pi = -\frac{1}{128}\sqrt{3}$; $\frac{3}{32}\sin\frac{4}{6}\pi = \frac{3}{64}\sqrt{3}$; $\frac{3}{16}\sin 2\theta = \frac{3}{32}\sqrt{3}$.

$$\therefore \int_0^{\frac{1}{6}\pi} \cos^3 3\theta \cos\theta d\theta = \tfrac{81}{640}\sqrt{3}.$$

The reader will have no difficulty in applying the same method to develop $\sin^n\theta$ and then for integrating $\sin^n\theta d\theta$.

It will be observed that when we put $\cos\theta = \frac{1}{2}(x + \frac{1}{x})$ we do not escape the impossible; for this is as much an impossible form as $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ for $x + \frac{1}{x}$ can never be *less* than 2, and $2\cos\theta$ can never be *greater* than 2.

CONCERNING CONICS THROUGH FOUR POINTS.

By EDGAR H. JOHNSON, Professor of Mathematics, Emory College, Oxford, Georgia.

The equation of the conic through $a_1 b_1$, $a_2 b_2$, $a_3 b_3$, $a_4 b_4$, and a fifth point $x_1 y_1$ is

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1{}^2 & x_1 y_1 & y_1 & x_1 & y_1 & 1 \\ a_1{}^2 & a_1 b_1 & b_1{}^2 & a_1 & b_1 & 1 \\ a_2{}^2 & a_2 b_2 & b_2{}^2 & a_2 & b_2 & 1 \\ a_3{}^2 & a_3 b_3 & b_3{}^2 & a_3 & b_3 & 1 \\ a_4{}^2 & a_4 b_4 & b_4{}^2 & a_4 & b_4 & 1 \end{vmatrix} = 0,$$

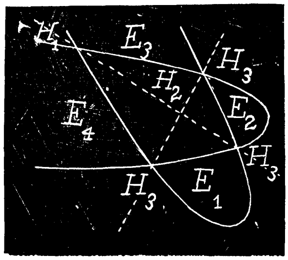
or $Ax^2 + 2Bxy + Cy^2 + 2Fx + 2Gy + H = 0$, where the coefficients A, B, C, ... are of the second degree in x_1 and y_1. The conic is an ellipse, parabola, or hy-

*Professor Waldo first called my attention to this easy method for integrating this particular expression.

perbola according as $AC-B^2$ is greater than, equal to, or less than zero. Through every point of the curve $AC-B^2=0$ (x_1 and y_1 being now general coordinates) may be drawn a parabola also passing through the four given points. Now it is known that through four points two parabolas can be drawn, the parabola being real or imaginary according as one of the four points does not or does lie in the triangle formed by the other three. (See Salmon's Conic Sections, page 153, ex. 1; or C. Smith's Conic Sections, pages 233-4).

Since through every point of each of these two parabolas, a parabola passing through the four given points is possible, the curve $AC-B^2=0$, of the fourth degree, decomposes into these same two parabolas.

Since $AC-B^2$ changes sign when a point crosses the curve, we have determined the locus of those points which with the four given points determine an ellipse (or hyperbola). The curve divides the plane into regions of two kinds, those for which $AC-B^2$ is positive, and those for which $AC-B^2$ is negative. Every point in a region of the first kind determines with the four given points an ellipse; every point of the second kind determines likewise an hyperbola. The points within the region enclosed by the two parabolas determine hyperbolas, since the four points determine a pair of straight lines, passing through this region, and for a pair of straight lines $AC-B^2<O$. Points in the regions marked H (see figure) determine with the four points of intersection of parabolas conics which are hyperbolas; points in the regions marked E determine likewise ellipses. A particular case of special interest arises when the four points become two pairs of coincident points, and the system becomes that of conics tangent to two given lines at given points. It is easy to show that the two parabolas become coincident. $AC-B^2$ is then a square and cannot change sign. The two tangents constitute one conic of the system and for the present purpose a pair of straight lines is a hyperbola. Hence all conics of the system, with the exceptions of the parabola and the pair of tangent lines, are hyperbolas.



In the above we have supposed points and conics to be real. It is easy to see that the condition for the passing of a real ellipse through four distinct real points is the same as for a real hyperbola. A real parabola can always be drawn through four real points not in the same straight line.

INTEGRAL SIDES OF RIGHT TRIANGLES.

By M. A. GRUBER, A. M., War Department, Washington, D. C.

$$a^2+b^2=c^2.$$

Problem I. To find integral sides of right triangles.

Rule 1. Take two integers, both odd or both even. ½ the sum of their squares equals the hypotenuse, or c; ½ the difference of their squares equals one of the legs, or b; and their product equals the other leg, or a.

Rule 2. Take any two integers. The sum of their squares equals the hypotenuse, or c; the difference of their squares equals one of the legs, or b; and twice their product equals the other leg, or a.

Rule 3. If *prime* integral sides are desired, the integers chosen must be prime to each other; in Rule 1, both odd; and in Rule 2, one odd and the other even.

Note. Rules 1 and 2 hold good also for fractional values. These rules are deduced from the two formulas mentioned in Problem II, and, to avoid repetition, are not discussed in this problem.

Problem II. Given one of the legs of a right triangle of integral sides to find the other leg and the hypotenuse.

The sides of a right triangle depend upon the equation $a^2+b^2=c^2$, in which a and b are the legs and c the hypotenuse of the triangle.

In the discussion of this problem, a is taken as the *given leg*.

When integral equations of the form $a^2+b^2=c^2$ are considered, the sets of values for a, b. and c are divided into two classes: (1) Those having no common factor; a, b, and c being prime integral values. (2) Those having a common factor; a, b, and c being found by multiplying a, b, and c of the first class by the highest common factor.

Sets of *prime* integral values are, therefore, the basis of work.

In right triangles of integral sides, any integer from 3 up may be taken as the value of one of the legs.

There are three kinds of integers to be considered: (1) Odd numbers; (2) Even numbers divisible by 4; and (3) Even numbers that are 2 times an odd number.

a may, then, be any one of these three kinds of numbers.

When a is *an odd number*, we have the formula

$$(mn)^2+\left(\frac{m^2-n^2}{2}\right)^2=\left(\frac{m^2+n^2}{2}\right)^2$$

by means of which to find b and c, so that a, b, and c have no common factor.

$$mn=a,\ \frac{m^2-n^2}{2}=b,\text{ and }\frac{m^2+n^2}{2}=c.$$

m and n are odd and are prime to each other, and $m>n$. There are as many

sets of prime integral values of a, b, and c as m and n can be made sets of odd, prime, integral factors, the product of each set of which factors equals a.

When a is *an even number divisible by* 4, we have the formula $(2mn)^2+(m^2-n^2)^2=(m^2+n^2)^2$, by means of which to find b and c, so that a, b, and c have no common factor. $2mn=a$, $m^2-n^2=b$, and $m^2+n^2=c$. m and n are prime to each other, one being odd, the other even; and $m>n$. There are as many sets of prime integral values of a, b, and c as m and n can be made sets of prime integral factors, the product of each set of which factors equals $\frac{1}{2}a$.

When a is *an even number that is* 2 *times an odd number*, we first find the set or sets of values for a equal to the odd number, and then multiply them by 2.

When a contains odd factors other than itself and unity, or even factors divisible by 4, there are other sets of values, in which a, b, and c have a common factor. There are as many sets of values of this kind as the sets of prime integral values that can be found for the odd factors and the even factors divisible by 4, cantained in a. In this case we first find the sets of prime integral values for each of the factors and then multiply them by the respective numbers that produce a.

In problems relating to the integral sides of right triangles, unity and the number itself are considered factors of a number.

For the purpose of bringing out the foregoing statements more clearly to the mind of the reader, we shall present them by way of illustration.

Put $a=3$, the lowest integer for integral sides of right triangles. Then $mn=3=3\times1$; whence $m=3$, $n=1$. Substituting these values in the formula for $a=$an odd number, we find $b=\frac{1}{2}(3^2-1^2)=4$, and $c=\frac{1}{2}(3^2+1^2)=5$. There is but one set of values; viz., 3, 4, 5.

Put $a=4$. Then $2mn=4=2\times2\times1$; whence $m=2$, $n=1$. Substituting these values in the formula for $a=$an even number divisible by 4, we find $b=2^2-1^2=3$, and $c=2^2+1^2=5$. This set of values, 4, 3, 5, is the same as that for $a=3$, only a and b have interchanged values. There is but one set.

Put $a=12$. Then $2mn=12=2\times6\times1$ and $2\times3\times2$. There are, therefore, two sets of prime integral values. To find first set, $m=6$, $n=1$. To find second set, $m=3$, $n=2$. Whence the sets are 12, 35, 37; and 12, 5, 13. But $12=4\times3$ and 3×4. Hence there are two other sets of values, each set having a common factor. When $a=3$, $b=4$, $c=5$. When $a=4$, $b=3$, $c=5$. Multiplying these sets by the respective numbers that produce $a=12$, we obtain the required sets, 12, 16, 20; and 12, 9, 15, making in all 4 sets.

Put $a=15$. Then $mn=15=15\times1$ and 5×3. There are, therefore, two sets of prime integral values: 15, 112, 113; and 15, 8, 17. But as $15=5\times3$ and 3×5, there are also two sets of values, each set having a common factor. When $a=3$, $b=4$, $c=5$. When $a=5$, $b=12$, $c=13$. Whence the required sets are 15, 20, 25; and 15, 60, 65,—in all 4 sets.

In order to find the number of sets of values that can be formed for $a=$an integer, we shall illustrate by taking $a=60$. Then $2mn=60=2\times30\times1$, $2\times15\times2$, $2\times10\times3$, and $2\times6\times5$. Hence there are 4 sets of prime integral val-

ues. But 60 contains also the following factors that are odd numbers: $3=3\times1$; $5=5\times1$; and $15=15\times1$ and 5×3. These give 4 more sets. The factors that are even numbers divisible by 4, are $4=2\times2\times1$; $12=2\times6\times1$ and $2\times3\times2$; and $20=2\times10\times1$ and $2\times5\times2$. These give 5 additional sets. Hence for $a=60$, there are 13 sets of values for integral sides of right triangles.

A THEOREM ON PRISMOID.

By P. H. PHILBRICK, C. E., Pineville, Louisiana.

THEOREM. *To prove that the error of the "end area volume" of any prismoid or solid to which the prismoidal formula applies, is twice the error of the "middle area volume" and on the opposite side of the true result.*

Let A and B represent the end areas, M the middle area, and l the length of the prismoid.

Then the true volume is, $V=\frac{1}{6}l(A+4M+B)$........................(1),

the end area volume is, $V_e=\frac{1}{2}l(A+B)$..................................(2),

and the middle area volume is, $V_m=lM$......................................(3).

Now (1)−(2) gives error of $(2)=V-V_e=\frac{1}{6}l(4M-2A-2B)$ ....(4),

and (1)−(3) gives error of $(3)=V-V_m=\frac{1}{6}l(A+B-2M)$..................(5).

But (4) is twice (5) with a contrary sign.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

74. Proposed by JOHN T. FAIRCHILD, Principal of Crawfis College, Crawfis College, Ohio.

When U. S. bonds are quoted in London at $108\frac{3}{4}$ and in Philadelphia at $112\frac{1}{2}$, exchange $\$4.89\frac{1}{2}$, gold quoted at 107, how much more was a \$1000 U. S. bond worth in London than in Philadelphia?

No solution of this problem has been received.

75. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

If 24 men, in 15 days of 12 hours each, dig a trench 300 yards long, 5 yards wide, 6 feet deep for 540 five-cent loaves when flour is \$8 a barrel; what is flour worth a barrel when 45 men, working $5\frac{1}{3}$ days of ten hours each, dig a trench 125 yards long, 5 yards wide, 8 feet deep for 320 four-cent loaves ? *Solve by proportion.*

Solution by G. B. M. ZERR, A. M., Ph. D., Tsxarkana, Arkansas, and C. A. JONES, Terrence, Miss.

The price of flour is an inverse ratio, hence using the *cause* and *effect* process we get at once

$$\left\{\begin{array}{r}24\\15\\12\\540\\5\end{array}\right\}:\left\{\begin{array}{c}300\\5\\6\\(?)\end{array}\right\}::\left\{\begin{array}{r}45\\5\frac{1}{3}\\10\\320\\4\end{array}\right\}:\left\{\begin{array}{c}125\\5\\8\\8\end{array}\right\}$$

$$\therefore\ (?)=\frac{24\times15\times12\times540\times5\times125\times5\times8\times8\times3}{300\times5\times6\times45\times16\times10\times320\times4}=16\tfrac{7}{8}.$$

$\therefore$ Flour is worth \$16.87½ per barrel.

76. Proposed by E. W. MORRELL, Professor of Mathematics in Montpelier Seminary, Montpelier, Vermont.

An eastern nobleman willed his entire estate to his three sons on the condition that the oldest should have one-half, the next one-third, and the youngest one-ninth. His estate, on inventory, was found to consist of 17 elephants. What should be the share of each ?

Solution by FREDERIC R. HONEY, Ph. B., New Haven, Connecticut, and CHAS. C. CROSS, Laytonsville, Maryland.

If the will was obeyed literally the eldest son's share was $\frac{17}{2}$ elephants ; the second son's $\frac{17}{3}$; and the youngest's $\frac{17}{9}$, making a total of $\frac{17}{2}+\frac{17}{3}+\frac{17}{9}=16\frac{1}{18}$ elephants. This would leave $\frac{17}{18}$ of an elephant.

The following solution would be satisfactory :

We have $\frac{1}{2}+\frac{1}{3}+\frac{1}{9}=\frac{17}{18}$ as the denominator.

First son receives $\dfrac{\frac{1}{2}}{\frac{17}{18}}=\frac{1}{2}\times\frac{18}{17}=\frac{9}{17}$.

Second son receives $\dfrac{\frac{1}{3}}{\frac{17}{18}}=\frac{1}{3}\times\frac{18}{17}=\frac{6}{17}$.

Third son receives $\dfrac{\frac{1}{9}}{\frac{17}{18}}=\frac{1}{9}\times\frac{18}{17}=\frac{2}{17}$.

Since the estate consisted of 17 elephants, $\therefore$ the first son got $\frac{9}{17}$ of $17=9$ elephants ; and the second son got $\frac{6}{17}$ of $17=6$ elephants ; and the third son got $\frac{2}{17}$ of $17=2$ elephants.

Remarks by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

As 17 is prime, the elephants should be divided as near the proportion as possible. $\therefore$ oldest should have 9, next, 6, and the youngest, 2.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

68. Proposed by LEONARD E. DICKSON, M. A., Ph. D., Formerly Fellow of Mathematics, University of Chicago, Chicago, Illinois.

Suppose a circle of unit radius divided at the points A, A_1, A_2, A_3, ... into n equal parts. [This division cannot in general be affected by geometry.] Through A draw the diameter OA and join O with A_1, A_2, A_3, ...$A_{\frac{n-1}{2}}$, where n is supposed to be odd.

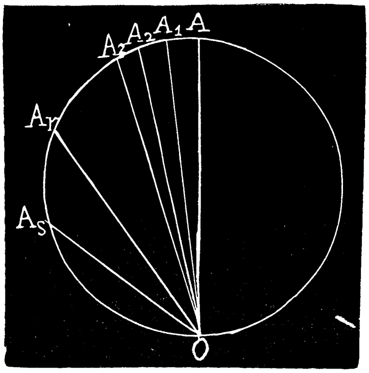
Prove that $OA_1 - OA_2 + OA_3 - OA_4 + \ldots\ldots \pm OA_{\frac{n-1}{2}}$, every other chord being affected with the minus sign.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas, and C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

Let OA_1, OA_2, OA_3, etc. $=a_1$, a_2, a_3, etc.

Now $OA=2$, $\angle AOA_1=\angle A_1OA_2=$ etc. $=\pi/n$.

$\therefore$ $OA_r=a_r=2\cos(r\pi/n)$.



(1) When $\frac{n-1}{2}$ is even,

$\therefore$ $a_1+a_3+a_5+\ldots\ldots+a_{\frac{n-3}{2}}$

$$=2\left(\cos\frac{\pi}{n}+\cos\frac{3\pi}{n}+\cos\frac{5\pi}{n}+\ldots\ldots\right.$$

$$\left.+\cos\frac{(n-3)\pi}{2n}\right)=\sin\frac{(n-1)\pi}{2n}\Big/\sin\frac{\pi}{n}$$

$$=\sin\left(\frac{\pi}{2}-\frac{\pi}{2n}\right)\Big/\sin\frac{\pi}{n}=\tfrac{1}{2}\cos\frac{\pi}{2n}\ \ldots\ldots\ (1).$$

$$a_2+a_4+a_6+\ldots\ldots+a_{\frac{n-1}{2}}=2\left(\cos\frac{2\pi}{n}+\cos\frac{4\pi}{n}+\cos\frac{6\pi}{n}+\ldots.+\cos\frac{(n-1)\pi}{2\pi}\right)$$

$$=\frac{2\cos[(n+3)\pi/4n]\sin[(n-1)\pi/4n]}{\sin(\pi/n)}=\frac{2\cos[\frac{1}{4}\pi+(3\pi/4n)]\sin[\frac{1}{4}\pi-(\pi/4n)]}{\sin(\pi/n)}$$

$$=\frac{[\cos(3\pi/4n)-\sin(3\pi/4n)][\cos(\pi/4n)-\sin(\pi/4n)]}{\sin(\pi/n)}=\frac{\cos(\pi/2n)-\sin(\pi/n)}{\sin(\pi/n)}$$

$$=\tfrac{1}{2}\mathrm{cosec}\frac{\pi}{2n}-1\ldots\ldots\ \ldots..(2).$$

$\therefore a_1-a_2+a_3-a_4+\ldots\ldots+a_{\frac{n-3}{2}}-a_{\frac{n-1}{2}}=1.$

(2) When $\frac{n-1}{2}$ is odd.

$$\therefore a_1+a_3+a_5+\ldots\ldots+a_{\frac{n-1}{2}}=2(\cos\frac{\pi}{n}+\cos\frac{3\pi}{n}+\cos\frac{5\pi}{n}+\ldots\ldots$$

$$+\cos\frac{(n-1)\pi}{2n})=\sin\frac{(n+1)\pi}{2n}/\sin\frac{\pi}{n}$$

$$=\sin\left(\frac{\pi}{2}+\frac{\pi}{2n}\right)/\sin\frac{\pi}{n}=\tfrac{1}{2}\text{cosec}\frac{\pi}{2n}\ldots\ldots\ldots\ldots\ (3).$$

$$a_2+a_4+a_6+\ldots\ldots+a_{\frac{n-3}{2}}=2\,(\cos\frac{2\pi}{n}+\cos\frac{4\pi}{n}+\cos\frac{6\pi}{n}+\ldots.+\cos\frac{(n-3)\pi}{2n})$$

$$=\frac{2\cos[(n+1)\pi/4n]\sin[(n-3)\pi/4n]}{\sin(\pi/n)}=\frac{2\cos[\frac{1}{4}\pi+(\pi/4n)]\sin[\frac{1}{4}\pi-(3\pi/4n)]}{\sin(\pi/n)}$$

$$+\frac{[\cos(n/4n)-\sin(\pi/4n)][\cos(3\pi/4n)-\sin(3\pi/4n)]}{\sin(\pi/n)}=\frac{\cos(\pi/2n)-\sin(\pi/n)}{\sin(\pi/n)}$$

$$=\tfrac{1}{2}\text{cosec}(\pi/2n)-1.\ \ldots\ldots\ldots\ldots(4).$$

$\therefore a_1-a_2+a_3-a_4+\ldots\ldots-a_{\frac{n-3}{2}}+a_{\frac{n-1}{2}}=1.$

$\therefore OA_1-OA_2+OA_3-OA_4+\ \ldots\ldots\pm OA_{\frac{n-1}{2}}=1.$

70. Proposed by **WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, Ohio.**

Prove that the locus of the center of the circle which passes through the vertex of a parabola and through its intersections with a normal chord is the parabola $2y^2=ax-a^2$, the equation to the given parabola being $y^2=4ax$.

Solution by the PROPOSER.

The circles being $(x-m)^2+(y-n)^2=r^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$,

and passing through the vertex of $y^2=4ax\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$,

becomes $x^2-2mx+y^2-2ny=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$.

Now the extremities of the normal chord being $(at_1{}^2,\ 2at_1)$, $(at_2{}^2,\ 2at_2)$, normal at the former point, we have

$$a^2t_1{}^4-2mt_1{}^2+4a^2t_1{}^2-4nat_1=0\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

$$\text{and } a^2t_2{}^4-2mt_2{}^2+4a^2t_2{}^2-4nat_2=0\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Divide these by at_1, at_2 respectively, and take one result from the other and divide by t_1-t_2; then

$$a(t_1{}^3+t_1{}^2t_2+t_1t_2{}^2)=2mt_1-4at_1 \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Divide (4) by at_1 and take (6) from the result; then

$$at_1t_2(t_1+t_2)=-4n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

But it can be shown that $t_1+t_2=-(2/t_1)$ .... (8).

Substituting in (7) and reducing, $t_2=(2n/a)=-t_1-(2/t_1)$ .... (9),

which gives $t_1=\dfrac{-n+\sqrt{n^2-2a^2}}{a}$ .... (10).

(10) in (6) gives $2n^2=am-a^2$ .... (11),

the required locus of center of (1).

Also solved by *F. M. McGAW*.

[NOTE. Solution of Problen 69 will appear in the next issue. Editor.]

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

44. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

There is a triangle whose sides repulse a center of force within the triangle with an intensity that varies inversely as the distance of the center of force from each point of the sides of the triangle. What the position of equilibrium of the center?

Solution by HENRY HEATON, M. Sc., Atlantic, Iowa.

Put p=altitude of the triangle upon the side a as base, s=distance of center of force from the side a, x=distance of any point of side a from the vertex B, y=distance of any point of the side b from the vertex C, and z=distance of any point of the side c from the vertex B.

The force exerted by any portion dx of the side a resolved perpendicular to it, is $msdx$ where m is an arbitrary constant depending on the intensity of the force. The forces exerted by portions dy and dz of the sides b and c are respectively $m[s-(b/p)y]dy$ and $m[s-(c/p)z]dz$. For equilibrium we have

$$m\int_0^a sdx+m\int_0^b (s-\frac{p}{b}y)dy+m\int_0^c (s-\frac{p}{c}z)dz=0.$$

$$\therefore\ (a+b+c)s=\tfrac{1}{2}p(b+c).\quad \therefore\ s=\frac{p(b+c)}{2\,(a+b+c)}.$$

Hence the distance of the center of force from the side a is $\frac{p(b+c)}{2(a+b+c)}$.

In like manner it may be shown that its distance from the side b is $\frac{q(a+c)}{2\,(a+b+c)}$, and that its distance from the side c is $\frac{r(a+b)}{2\,(a+b+c)}$, when q and r are the altitudes of the triangle upon the sides b and c respectively.

Also solved by *G. B. M. ZERR*. His solution will appear in the next issue.

45. Proposed by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

A fifty-pound cannon-ball is projected vertically upward with a velocity of 300 feet per second. Find the height to which it will rise and the time of flight, assuming the initial resistance of the air on the ball to be 10 pounds and the resistance to vary as the square of the velocity.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Let h=height required, t=time of ascent, t_1=time of descent, $T=t+t_1$= time of flight, v=velocity=300 feet per second. W=50 pounds, g=32.2 feet per second. $\mu v^2=10$ pounds$=\frac{1}{5}W$.

$$\therefore\ \mu=\frac{W}{5v^2}.\quad \frac{1}{k}=\sqrt{\frac{W}{\mu}}=v\sqrt{5}.\quad \therefore\ k=\frac{1}{v\sqrt{5}}.$$

From Bowser's Analytical Mechanics, pages 306-7, we get,

$$h=\frac{1}{2gk^2}\log(1+k^2v^2),\quad t=\frac{1}{gk}\tan^{-1}vk.$$

$$t_1=\frac{1}{gk}\log\{\sqrt{1+v^2k^2}+vk\}.$$

$$\therefore\ h=\frac{5v^2}{2g}\log(\tfrac{6}{5})=1273.9827 \text{ feet}.$$

$$t=\frac{v\sqrt{5}}{g}\tan^{-1}\frac{1}{\sqrt{5}}=8.75931 \text{ seconds}.$$

$$t_1=\frac{v\sqrt{5}}{g}\log\left(\frac{\sqrt{6}+1}{\sqrt{5}}\right)=9.03118 \text{ seconds}.$$

$$T=t+t_1=17.79049 \text{ seconds}.$$

Also solved by *HENRY HEATON*.

$t=355287708.316$ seconds$=11$ years, 3 months, 7 days, 3 hours, 1 minute, 48.316 seconds.

Also solved by *J. C. CORBIN.*

47. **Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in Columbian University, Washington, D. C.**

What is the focus of the convex surface of a plano-convex lens, index μ, which will converge parallel monochromatic rays to a given focus, the rays entering the lens on the plane side?

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Let f=the given focal length. F=the focal length required,

u=distance of origin of ray from lense,

r, s, the radii of the first and second surfaces of the lense respectively,

t=the thickness, and regard all distances as measured from the posterior surface.

Then we have for for a double convex lense,

$$\frac{1}{\frac{1}{f}+\frac{\mu-1}{s}}-\frac{1}{\frac{1}{u}-\frac{\mu-1}{r}}=\frac{t}{\mu}.$$

(See Parkinson's Optics, Art. 100, Cor. I, page 91).

Let $u=r=\infty$.

$$\therefore \frac{1}{\frac{1}{f}+\frac{\mu-1}{s}}=\frac{t}{\mu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

This is the plano-convex lense with light incident upon plane surface.

Write F for f, and let $s=u=\infty$.

$$\therefore \frac{1}{\frac{1}{F}}+\frac{1}{\frac{\mu-1}{r}}=\frac{t}{\mu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

This is the plano-convex lense with light incident upon the convex surface. Since we are using the same lense, $r=s$.

$$\therefore\ r=z=\frac{(\mu-1)ft}{\mu f-t}, \text{ from (1).}$$

This value of r in (2) gives, $F=-\dfrac{t^2(\mu-1)}{(\mu f-t)^2}$.

$\therefore$ F is found independent of the radius of convexity.

48. **Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.**

Two equal heavy rings connected by a string passing over a peg at the focus of a conic section will be in equilibrium at all points on the curve.

Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

An evident property of *any* curve which will be a curve of equilibrium for two weights thus attached is that the tension along the string shall be the same, wherever the weights are placed upon the curve. If this were not so, by altering the position of one of them, by changing the length of the string, the tension would be changed and the other would no longer be in equilibrium.

Call T the tension, W the weight, ϕ the angle which the curve makes with the horizontal, and θ the angle which the string makes.

Resolving along the curve,

$$W\cos\phi - T\cos(\theta-\phi)=0.$$

$$\frac{\cos(\theta-\phi)}{\cos\phi}=\frac{W}{T}=k.$$

$$\therefore\ \cos\theta+\sin\theta\tan\phi=k.$$

$$\frac{x}{\sqrt{x^2+y^2}}+\frac{y}{\sqrt{x^2+y^2}}\cdot\frac{dy}{dx}=k. \qquad \frac{xdx+ydy}{\sqrt{x^2+y^2}}=kdx.$$

$$(x^2+y^2)^{\frac{1}{2}}=kx+c. \qquad x^2(1-k^2)+y^2-2kcx=c^2.$$

This is the equation of a conic with the origin at the focus.

$$k=e+c=a(1-e^2).$$

The above investigation refers to the case in which the tension is simply constant. The string may be attached to the fixed point.

If the string be now considered passing around the focus to the curve again and a weight W attached there also, the tension will be doubled.

Then $k=W/2T$.

If $W=2T$, or $T=\frac{1}{2}W$, the equation becomes $y^2-2cx=c^2$, a parabola.

If $W>2T$, or $T<\frac{1}{2}W$, the curve is an hyperbola.

If $W<2T$, or $T>\frac{1}{2}W$, the curve is an ellipse.

The above may be put in the following form :

In a parabola the tension is equal to half one of the equal weights ; in an hyperbola it is less than half of the weight ; and in an ellipse it is greater than the same.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

43. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

In a circle whose radius is a, chords are drawn through a point distant b from the center. What is the average length of such chords, (1), if a chord is drawn from every point of the circumference, and (2), if they are drawn through the point at equal angular intervals?

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Let $AC=CF=a$, $CD=b$, $\angle FDA=\theta$, $\angle FCA=\phi$, $\tan\theta=m$.

Then the equation to DF is, $y=mx+mb$ ....(1).

The equation to the circle is, $x^2+y^2=a^2$ ....(2).

From (1) and (2) we easily get $EF=2\sqrt{a^2-\dfrac{m^2b^2}{1+m^2}}$,

$\therefore$ $EF=2\sqrt{a^2-b^2\sin^2\theta}$. But $\sin\theta=\dfrac{a\sin\phi}{\sqrt{a^2+b^2+2ab\cos\phi}}$.

$$\therefore\ EF=\frac{2(a^2+ab\cos\phi)}{\sqrt{a^2+b^2+2ab\cos\phi}}=\frac{2(a^2+ab-2ab\sin^2\frac{1}{2}\phi)}{\sqrt{(a+b)^2-4ab\sin^2\frac{1}{2}\phi)}}.$$

The limits of ϕ for $b>a$, are 0 and $\frac{1}{2}\pi+\sin^{-1}(a/b)=2\beta$.

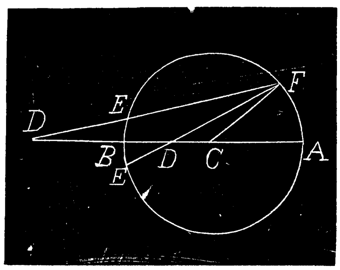
The limits of ϕ for $b<a$, are 0 and $\frac{1}{2}\pi+\sin^{-1}(b/a)=2\delta$.

The limits of θ for $b>a$, are 0 and $\sin^{-1}(a/b)=\theta'$.

The limits of θ for $b<a$, are 0 and $\frac{1}{2}\pi$.

Let Δ and Δ_1 be the average lengths required.

$$\therefore\ \Delta=\int EFds/\int ds=\int EFd\phi\,/\int d\phi,$$



$$\Delta_1=\int EFd\theta/\int d\theta.$$

I. Let $\dfrac{4ab}{(a+b)^2}=e^2$, and $\frac{1}{2}\phi=\gamma$.

$\therefore$ $d\phi=2d\gamma$.

Then $EF=\dfrac{2(a^2+ab-2ab\sin^2\gamma)}{(a+b)\sqrt{1-e^2\sin^2\gamma}}$.

$$\therefore\ EF=\frac{2a}{\sqrt{1-e^2\sin^2\gamma}}+\frac{2}{e\sqrt{ab}}\sqrt{1-e^2\sin^2\gamma}-\frac{2}{e\sqrt{ab}\sqrt{1-e^2\sin^2\gamma}}.$$

$$\therefore \int EFd\gamma=\frac{2}{e\sqrt{ab}}\{(ae\sqrt{ab}-1)F(e,\ \gamma)+E(e,\ \gamma)\}.$$

$$\therefore\ \Delta=\frac{2}{\beta e\sqrt{ab}}\{(ae\sqrt{ab}-1)F_0^{\beta}(e,\ \gamma)+E_0^{\beta}(e,\ \gamma)\},\ b>a.$$

$$\Delta=\frac{2}{\delta e\sqrt{ab}}\{(ae\sqrt{ab}-1)F_0^{\delta}(e,\ \gamma)+E_0^{\delta}(e,\ \gamma)\},\ b<a.$$

II. $\Delta_1=2\int_0^{\theta'}\sqrt{a^2-b^2\sin^2\theta}\,d\theta\Big/\int_0^{\theta'}d\theta=\dfrac{2}{\theta'}\int_0^{\theta'}\sqrt{a^2-b^2\sin^2\theta}d\theta.$

Let $\theta=\frac{1}{2}\pi+\lambda$. $\therefore\ \theta'-\frac{1}{2}\pi=\lambda$ to $-\frac{1}{2}\pi=\lambda$.

$$\therefore\ \Delta_1=\frac{2}{\theta'}\int_{-\frac{1}{2}\pi}^{\theta'-\frac{1}{2}\pi}\sqrt{a^2-b^2\cos^2\lambda}\,d\lambda=\frac{2}{\theta'}\int_{-\frac{1}{2}\pi}^{\theta'-\frac{1}{2}\pi}\sqrt{b^2\sin^2\lambda-(b^2-a^2)}d\lambda$$

$$=\frac{2\sqrt{(b^2-a^2)}}{\theta'}\int_{-\frac{1}{2}\pi}^{\theta'-\frac{1}{2}\pi}\sqrt{\frac{b^2}{b^2-a^2}\sin^2\lambda-1}\,d\lambda$$

$$=\frac{2\sqrt{(b^2-a^2)}}{\theta'}H_{-\frac{1}{2}\pi}^{\theta'-\frac{1}{2}\pi}\left(\frac{b}{\sqrt{b^2-a^2}},\ \lambda\right),\ b>a.$$

$$\Delta_1=2\int_0^{\frac{1}{2}\pi}\sqrt{a^2-b^2\sin^2\theta}\,d\theta\Big/\int_0^{\frac{1}{2}\pi}d\theta=\frac{4a}{\pi}E_0^{\frac{1}{2}\pi}\left(\frac{b}{a},\ \theta\right),\ b<a.$$

45. Proposed by **J. C. WILLIAMS**, Boston, Massachusetts.

At the end of the fifth inning the base ball score stands 7 to 9. What is the probability of winning for either team?

Solution by **J. SCHEFFER, A. M.**, Hagerstown, Maryland.

From the stated score we are able to estimate the respective skill of the two teams, and their respective probabilities of winning the game.

The respective probabilities are $\frac{7}{16}$ and $\frac{9}{16}$. We have now to find the probabilities of either team winning *at least* 3 games out of 4, granting, of course, 9

innings to be played. These probabilities are respectfully, $(\frac{7}{16})^4+4(\frac{7}{16})^3\cdot\frac{9}{16}=\frac{14749}{65536}$, and $(\frac{9}{16})^4+4.(\frac{9}{16})^3\cdot\frac{7}{16}=\frac{26973}{65536}$.

46. **Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.**

Four men starting from random points on the circumference of a circular field and traveling at different rates, take random straight courses across it; find the chance that at least two of them will meet.

Professor Heaton says : "If the men are considered points the chance is 0." [A possible though difficult problem could be made of this one by using instead of men segments of straight lines moving along random secants of a circle, the velocity of the segments all being different. Editor.]

47. **Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.**

What is the average length of the chords that may be drawn from one extremity of the major axis of an ellipse to every point of the curve ?

Solution by the PROPOSER.

The length of a single chord is

$$[(a-x)^2+y^2]^{\frac{1}{2}}=(1/a)[a^2(a^2-x^2)+b^2(a^2-x^2)]^{\frac{1}{2}}\ .$$

Put S=distance around the ellipse. Then the required average is $A=$

$$\frac{2}{aS}\int_0^{\frac{1}{2}S}[a^2(a-x)^2-b^2(a^2-x^2)]^{\frac{1}{2}}\,dS=$$

$$\frac{2}{a^2S}\int_{-a}^{+a}\frac{[a^2(a-x)^2+b^2(a^2-x^2)]^{\frac{1}{2}}\,[a^2(a^2-x^2)+b^2x^2]^{\frac{1}{2}}\,dx}{(a^2-x^2)^{\frac{1}{2}}}=$$

$$\frac{2}{a^2S}\int_{-a}^{+a}\frac{[a(a^2+b^2)-(a^2-b^2)x]^{\frac{1}{2}}\,[a^4-(a^2-b^2)x^2]^{\frac{1}{2}}\,dx}{(a+x)^{\frac{1}{2}}}.$$

This is readily reducible to elliptic functions of the first and second order, but the expressions I have been able to obtain are involved radicals.

Also solved by *G. B. M. ZERR* and *J. F. SCHEFFER.*

NOTE ON PROBLEM 39I

BY LEWIS NEIKIRK, BOULDER, COLORADO.

The man starts at O moving in a *perfectly random* manner. After t seconds suppose him at P and that during the next instant dt he travels through ds to m at an angle θ with the line OP. Let $PM=dr=ds\cos\theta=v\cos\theta dt$, since $ds=vdt$. He will escape from the desert if $\int dr>R$ (the radius) the limits of inte-

gration being those which correspond to 0 and T of t; that is, if $\int_0^T v\cos\theta dt > R$. But this integral depends upon two independent variables. Indeed, θ, being wholly discontinuous from point to point according to the conditions of the problem, can not be considered a variable at all. If however, we assume θ constant (i. e. if the "perfectly random" motion of the problem means motion in a logarithmic spiral) then the condition above reduces to $vT\cos\theta > R$; or $\theta > \cos^{-1}(R/vT)$, agreeing with Professor Anthony.

NOTE ON PROBLEM 39.

BY J. BURKETT WEBB, C. E., PROFESSOR OF MATHEMATICS AND MECHANICS, STEVENS INSTITUTE OF TECHNOLOGY, HOBOKEN, NEW JERSEY.

It seems to me that every such problem should have a complete and intelligible *physical* idea behind it, and further that a solution should be a development of the *physical ideas* of the problem, mathematics being simply the grammatical language of physics.

If Professor Anthony has a complete idea in the problem it is not intelligible to me and so it may be best to state the difficulties which appear to me.

It is to be inferred from the solution that the "perfectly random manner" means that the path consists of differential elements of equal length and all possible directions arranged in a chance succession.

If so the man will never reach the edge of the desert, or, stated otherwise, he will have but one chance in an infinite number of doing so.

In the solution the *rate of approach to the circumference* is spoken of; in random movements there would be no such rate except as the average of actual rates and this is not the use made of it.

The solution also supposes the man at each instant to go *within the angle* MPK, but this he does not need to do to get off in the time; so the deduced chance seems not to follow.

In fact the chance C=etc., is the answer to a different problem, as I see the matter, namely: Of all logarithmic spirals joining the center and circumference, having their origins at the center of the circle and differing from each other by equal increments of the angle between the radius vector and curve, what is the chance of choosing at random one whose included arc shall be less than Tv?

To make the problem apply to the case, for which it was I suppose, intended, of a wanderer in a desert I think one of two things will be needed. Either a certain finite length of step, taken at random must be fixed, or a law established to make large changes of direction less likely than small ones.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

77. Proposed by F. S. ELDER, Professor of Mathematics, Oklahoma University, Norman, Oklahoma.

For how many seconds must I count the clicking of the rails under a train that the number of rails counted may be equal to the speed of the train in miles per hour, a rail being 30 feet long?

78. Proposed by NELSON S. RORAY, South Jersey Institute, Bridgeton, New Jersey.

Solve by pure arithmetic, no algebraic symbols: A Texan farmer owns 5169 cattle; there are 3 times as many horses as cows, plus 569, and 4 times as many cows as sheep, minus 126; how many has he of each? [From *Brooks' Higher Arithmetic.*]

79. Proposed by F. M. PRIEST, St. Louis, Missouri.

How many $20 gold pieces can be put in a room 20 feet long, 18 feet wide, and 9 feet high?

GEOMETRY.

77. Proposed by CHARLES C. CROSS, Laytonsville, Maryland.

A line is drawn perpendicular to BC, of the triangle ABC, whose sides are $BC=a$, $CA=b$, and $AB=c$, through A to D, a distance d, (d being equal to or greater than $a+b$); from D a line is drawn to E, a distance e, (e being equal to or greater than $a+b+c$) on BC extended. Required the area of the ellipse which is isogonal conjugate to the straight line DE with respect to the triangle ABC.

78. Proposed by J. A. MOORE, Professor of Mathematics, Millsaps College, Jackson, Mississippi.

Required the number of normals that can be drawn from any point (a, b) to the parabola $y^2=2px$.

77. Proposed by JOHN MACNIE, Professor of Mathematics, University of North Dakota, University, North Dakota.

To construct a quadrilateral of given area, the diagonals, one of which is given, cutting each other in given ratios and at a given angle.

MECHANICS.

55. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University P. O., Mississippi.

Three equal heavy spheres, each of weight W, are placed on a rough ground just not touching each other. A fourth sphere of weight nW is placed on the top touching all three. Show that there is equilibrium if the coefficient of friction between two spheres is greater than $\tan\frac{1}{2}\alpha$, and that between a sphere and the ground is greater than $\tan\frac{1}{2}\alpha n/(n+3)$, where α is the inclination to the vertical of the straight line joining the centers of the upper and one lower sphere.

56. Proposed by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

"Hey-diddle-diddle, the cat and the fiddle,
The cow jumped over the moon."

Taking the weight of the cow to be 600 pounds, the initial resistance of the air to be 100 pounds and varying as the square of the velocity, find the initial and final velocities and the times of rising and falling.

57. **Proposed b J. C. NAGLE, M. A., M. C. E., Professor of Civil Engineering, Agricultural and Mechanical College of Texas, College Station, Texas.**

Over the intersection of two inclined planes slides a cord of uniform mass throughout its length. Find the equation to the path described by its center of gravity.

AVERAGE AND PROBABILITY.

54. **Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.**

A man is at the center of a circle whose diameter is equal to three of his steps. If each step is taken in a perfectly random direction, what is the probability, (1), that he will step outside the circle at the second step, and, (2), that he will step outside at the third step?

55. **Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.**

It has been clear for 15 consecutive days, what is the chance of the 16th day being cloudy?

56. **Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.**

Find the chance that the center of gravity of a triangle lies inside the triangle formed by three points taken at random within the triangle. [From *Williamson's Integral Calculus.*]

NOTES.

NOTE ON MR. BECHER'S ARTICLE IN OCTOBER NUMBER OF MONTHLY.

BY J. R. BALDWIN, DAVENPORT, IOWA.

In Franklin A. Becher's article for the October number, (Vol. III), I notice he says, "Multiplying an infinite number by another gives us infinity of a higher power, or dividing gives us infinity of a lower power."

How does he reconcile the latter part of this statement with Wallis's expression for the value of π,

$$\tfrac{1}{2}\pi=\frac{2.2.4.4.6.6.8.8\ \ldots\ldots}{1.3.3.5.5.7.7.9\ \ldots\ldots}\ ?$$

In this expression, we have the quotient of two infinite numbers equal to a finite number.

Mr. Lilley's criticism (MONTHLY, Vol. III., No. 3.) of the solution IV. (MONTHLY, Vol. II., page 190) is undoubtedly valid, but the statement that "Todhunter failed to produce a direct proof of it" is probably incorrect. The theorem is given by Todhunter (Euclid, page 316) and in a note at the bottom of page 317 he says, "For the history of this theorem see *Lady's and Gentlemen's Diary* for 1859, page 88." If any reader of the MONTHLY has the Diary for that year I should be very much pleased to see the history of this theorem published in the MONTHLY. Todhunter's proof of the theorem is indirect, but that does not argue that he was unable to discover a direct proof. I remember that many

years ago when reading Todhunter's Euclid I attempted a direct proof of this theorem but failed. The proof on page 157, Vol. II. of the MONTHLY is a direct proof and, with the exception of a few mistakes in lettering, seems to be free from objection. A slight simplification may be made by proving the equality of the triangles ADB, BFA, instead ADF, BDF.

WM. E. HEAL.

MULTI-DIRECTIONAL GEOMETRY.

BY JOHN N. LYLE, PH. D., BENTONVILLE, ARKANSAS.

The concept plane, rectilineal angle implies that there are straight lines and also, that they are located in different directions.

Hence, no system of plane geometry or of spherical geometry for that matter, is free from assumptions regarding "direction."

In some geometrical systems, however, larger use is made of "direction," both word and thing, than in others. Euclid, by his three geometrical axioms and his three postulates places restriction upon "directional geometry" which cannot be relaxed without endangering these axioms and postulates.

According to the Euclidean geometry there is but *one straight path* from the point A to the point B. That path marks the direction from A to B. A body moving in this direction on this path approaches B until B is reached. A body moving in the opposite direction along the same straight path recedes farther and farther from B.

This is the pure Euclidean doctrine, clear and strong, free from the suspicion even of a hypothetical "point at infinity" where ungeometrical deeds are reported to be done.

According to the Euclidean view, then, there is but one direction from A to B. According to Olans Henrici's view as given in the Article on Geometry in the Ninth Edition of the Encyclopœdia Britannica there are at least two directions diametrically opposite to each other from A to B; one direct and finite in length; the other roundabout via "the point at infinity."

This latter route can hardly be called "air line." Let us notice just one logical difficulty. Every path that reaches B drawn from A must be continuous. But a continuous line with two ends A and B must be finite. Hence, the hypothesis that a continuous line, infinite in length, can be drawn between two points A and B is a flagrant violation of the logical law of Non-Contradiction. By the way, this logical law is the bed rock on which the *reductio ad absurdum* process of reasoning is founded.

Another species under the genus Directional hypothesis is the Multi-Directional hypothesis. According to this hypothesis B may be so located with respect to A that myriads of different straight lines may be drawn between the two points. That is, B is myriads of different directions from A. This result seems to me to be absurd. Hence, for that reason, I would reject it. Many modern mathematicians, however, regard their hypotheses as beyond the reach of *reductio ad absurdum* method and the fundamental laws of thought.

EDITORIALS.

Professor Colaw was called away from home during the greater part of the past month, which fact will explain why his departments have been omitted in this issue.

We are happy to announce that a series of short elementary expository articles on Lie's Transformation Groups by Dr. E. O. Lovett, Baltimore, Maryland, will begin in the May number.

Professor Ollis Howard Kendall died last week at his home in Philadelphia. Professor Kendall was for a number of years Assistant Professor of Mathematics at the University of Pennsylvania, at the same time that his father occupied the Chair of Mathematics.

BOOKS AND PERIODICALS.

Algebra Reviews. By Edward Rutledge Robbins, Master in Mathematics and Physics, The Lawrenceville School. Paper Back, 44 pages. Chicago: Ginn & Co.

The object of this little book is to present the essentials of Elementary Algebra in a form sufficiently complete as to be helpful to teachers and students at the time of review. The exercises are various and well selected. Teachers desiring such a book, will find this one well suited to their needs. B. F. F.

Thoughts on Religion. By the late George John Romanes, M. A., LL. D., F. R. S., Canon of Westminster. Edited by Charles Gore, M. A., Canon of Westminster. Cloth, gilt top, 184 pages. Price, $1.25. Chicago: The Open Court Publishing Co.

The value and importance of this work on the thought and conscience of the world cannot be overestimated. Coming as it does from one of the foremost agnostics and scientific thinkers of his time, it comes as a revelation to all classes of readers. In this book can be studied the evolution of a master mind from adhering to the doctrine of agnosticism to that of a full acceptance of the religion of Jesus Christ. B. F. F.

Darwin, and After Darwin. An Exposition of the Darwinian Theory and a Discussion of the Post-Darwinian Questions. By George John Romanes, M. A., LL. D., F. R. S., Honorary Fellow of Gonville and Caius College, Cambridge. I. The Darwinian Theory. Second Edition. Cloth, gilt top, xiv and 460 pages. Price, $2.00. Chicago: The Open Court Publishing Co.

The first volume contains ten chapters. Chapter I, Introductory; Chapter II, Classification; Chapter III, Morphology; Chapter IV, Embryology; Chapter V, Palæontology; Chapter VI, Geographical Distribution; Chapter VII, The Theory of Natural Selection;

Chapter VIII, Evidences of the Theory of Natural Selection; Chapter IX, Criticisms of the Theory of Natural Selection; Chapter X, The Theory of Sexual Selection, and concluding remarks. A more earnest and convincing argument in favor of the Theory of Evolution has not appeared since Darwin's time. Dr. Romanes' grasp of thought and power of cogent reasoning appears in this volume with telling effect. No one with a fair knowledge of the methods of scientific investigations can fail, after having read this book, to be convinced of the truth of the theory. B. F. F.

University Algebra. By C. A. Van Velzer and Chas. S. Slichter, Professors in the University of Wisconsin. Pages 732. Madison, Wisconsin: Tracy, Gibbs and Company. 1893.

This book is now too well known to need any commendation from us. The authors are able and progressive teachers and in this text on algebra have introduced several new and valuable features. There are valuable chapters on mathematical induction, theory of limits, derivatives, complex numbers, the rational integral function, special equations, separation of roots, numerical equations, decomposition of rational fractions, graphic representation of equations, and determinants. The convergence and divergence of series is admirably treated. The accurate "historical notes" which are appended to the treatment of many of the topics will be appreciated. Every teacher of algebra has need of this work in his library whether he uses it as a class text-book or not. J. M. C.

Text-Book of Dynamics. University Tutorial Series. By William Briggs, M. A., F. C. S., F. R. A. S., and G. H. Bryan, M. A. Cloth, 105 pages. Price, 50 cents. Cambridge, England: W. B. Clive. New York Depot: Hinds & Noble, 4 Cooper Institute.

We called attention in a previous number to the text-book on *Hydro-Statics* by the same authors. The treatise on *Dynamics* deserves the same commendation. Due prominence is given to the principles of the subject, and in the solution of problems results are deduced as far as possible from these principles themselves. Worked examples are freely inserted, and hints relating to special difficulties are given where needed. The examples are numerous and practical, the examination papers well selected, and the summary of results after each chapter of special value in reviews. The book may be open to criticism on some minor points, but there are few text-books on this subject which are so well suited to the needs of beginners. J. M. C.

Theoretical Mechanics: Fluids. By J. Edward Taylor, M. A., B. Sc. 222 pages. Price, 80 cents. London and New York: Longmans, Green & Co.

Although intended to meet the Science and Art Department and London Matriculation requirements, this book may be used successfully in any school where a good text-book of its grade is required. One of the special features of the book is the large number of model examples which are fully worked out. The author believes they serve to fix the subject matter on the mind much more than simply reading over the text. This feature also makes it a valuable book to private students. The text is supplied with numerous graduated examples. J. M. C.

A Treatics on Elementary Hydrostatics. By John Greaves, M. A. Price, $1.10. 204 pages. Cambridge Press. New York: Macmillan & Co.

The author aims to treat the subject as fully as possible without using the Calculus, except in alternative proofs when by its aid results are more easily obtained or more concisely expressed. The mathematical element of the book is strong, and the book more advanced than the title and proposed method of treatment would indicate. It is well printed and furnished with sets of carefully selected exercises, while there are many excellent illustrative solutions. The topical index is helpful for ready reference. J. M. C.

Geometry of the Similar Figures and the Plane. By C. W. C. Barlow, M. A., B. Sc., and G. H. Bryan, M. A. Price, 60 cents. 123 pages. University Tutorial Series. Cambridge: W. B. Clive. New York Depot: Hinds & Noble.

This little book contains the Sixth and Eleventh Books of Euclid, together with a summary of Book V., and many important additional propositions and applications relating to the Geometry of Similar Figures and the Plane. Euclid's order has been closely followed, while the additional matter is mostly in the form of illustrative examples. The properties of centers of similitude and homologous points are collected in a supplement at the end of Book VI. In addition to the illustrative examples, numerous exercises for solution follow the propositions on which they depend. The feature of giving many alternative proofs enables the teacher to make his own choice of methods. It is a very satisfactory book in a useful series. J. M. C.

Modern Plane Geometry. By G. Richardson, M. A., and A. S. Ramsay, M. A. Price, $1.00. 202 pages. London and New York: Macmillan & Co.

This treatise includes chapters on properties of a triangle, quadrangle, and circle, harmonic and anharmonic ratio, geometrical maxima and minima, involution, reciprocation, inversion, and projection. . It gives all that is best in the recent geometry on these subjects and is an excellent introduction to the more advanced books of Cremona and others. In arrangement the sequence of propositions recommended by the Association for the improvement of Geometrical Teaching has been followed. The triangle has been very fully and satisfactorily treated. The book will serve as an excellent sequel to Euclid, and as a means of proceedure from Euclidean Geometry to the higher descriptive Geometry of Conics and of imaginary points. J. M. C.

Our Notions of Number and Space. By Herbert Nichols, Ph. D., assisted by W. E. Parsons, A. B. 201 pages. Price, $1.00. Ginn & Company, Boston.

This book is an experimental contribution to the "Genetic Theory of Mind." It aims to trace out the origin and development of our present perceptions of number and space from the nature of our past experiences. The experiments were conducted with great care and patience, and the results are worthy of being placed in this permanent and accessable form. The general survey and summary at the end of the book are helpful and valuable. J. M. C.

Business Forms, Customs, and Accounts. By Seymour Eaton. Price of Exercise Manual, 50 cents; price of Book of Forms, $1.00. American Book Company, New York and Chicago.

This manual provides a course of instruction in business which may be used to advantage in schools of all grades where the principles of business are taught. The principles of double entry bookkeeping are taught, but the application of principles to the needs of each particular business are left to be learned in that business. The work is planned to encourage original effort. The exercises are drawn largely from actual transactions. The questions are practical and suggestive. The excellent Book of Forms which accompanies the Exercise Manual will serve to make the teaching of this study both easy and effective. J. M. C.

Spencerian System of Penmanship: Common School Course. No. 10, "Connected Business Forms;" No. 11, "Double Entry Bookkeeping." Price, 8 cents each. American Book Company, New York and Chicago.

These books afford the pupil exercise in penmanship, and also familiarize him in a practical manner with ordinary business forms. J. M. C.

Patriotic Citizenship. By Thomas J. Morgan, LL. D. Price, $1.00. 368 pages. American Book Company, New York, Cincinnate, and Chicago. 1895.

The method of this book is a catechism of about 140 questions with as many concise and comprehensive answers by the author. The text of the answers is followed by brief citations from a wide range of authorities chiefly American. Here is found collected much of the finest literature on the selected topics, so arranged as to explain and enforce the text. The book is designed primarily for the public schools following a course in U. S. history, but it is also a good book for the citizen, reading circle, or family. We think the book lacks some of the helps, in the way of outline, contents or index, showing the relation of selected topics to central theme, etc., which would have secured better adaptation from a teaching point of view. The study of this book will give good results in stimulating patriotism and promoting good citizenship. J. M. C.

Elementary Lessons in Algebra. By Stewart B. Sabin and Charles D. Lowry. Price, 50 cents. 128 pages. New York, Cincinnati, and Chicago: American Book Company.

This little book was prepared to meet the demand for a text-book exactly suited to introduce the study of Algebra into Grammar Schools. The development is inductive, and in arrangement, method, problems and exercises, it is well adapted for its purpose. J. M. C.

Elements of Plane Geometry. By John Macnie, A. M., author of "Theory of Equations." Edited by Emerson E. White, A. M., LL. D., author of "White's Series of Mathematics." Price, 75 cents. 240 pages. 1895. New York, Cincinnati, and Chicago: American Book Company.

In this edition the Plane Geometry is bound separately. We reviewed the Plane and Solid Geometry as bound together in our issue of June, 1895, and a further examination gives us no reason to withdraw the favorable comments made on this book in that notice. J. M. C.

Inductive Studies in English Grammar. By William R. Harper, Ph. D., President of the University of Chicago, and Isaac M. Burgess, A. M., Professor in the University of Chicago. Cloth, 12mo, 96 pages. Price, 40 cents. New York, Cincinnati, and Chicago: American Book Company.

This book presents in a brief compass a systematic course in English Grammar, with special reference to its relation and analogy to other languages. The essential facts of the language are briefly and concisely stated, while the terminology and method of presentation are more closely adapted to that used in Latin Grammars. The pupil's knowledge is tested by requiring him to pick out concrete examples of its application from selections of connected English, instead of giving rules with classified groups of examples. The book is scholarly and has many strong points, and is excellently adapted for a review course in English preparatory to the study of the Ancient or Modern Languages. We believe it will meet with a wider use for this purpose and as supplemental to other grammars than as an independent class-book. J. M. C.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year in advance. Single copies, 10 cents. Irvington-on-the-Hudson, New York.

What is probably the most important discussion of the educational question ever held, has been opened in the April *Cosmopolitan.* President Gilman of the Johns Hopkins

University will follow the introductory article, and the leading educators of the day will contribute articles upon this most important inquiry: "Does Modern Education Educate, in the Broadest and Most Liberal Sense of the Term?" Those interested in the instruction of youth, either as teacher or parent, can not afford to miss this remarkable symposium, intended to review the mistakes of the nineteenth century, and signalize the entrance of the twentieth by advancing the cause of education. President Dwight of Yale, Presi- Schurman of Cornell, Bishop Potter and President Morton are among those who have already agreed to contribute to what promises to be the most significant series of educational papers ever printed. The aim is to consider existing methods in the light of the requirements of the life of to-day, and this work has never been undertaken on a scale in any degree approaching that outlined for *The Cosmopolitan*. Write to us for subscriptions.

B. F. F.

The Arena. An Illustrated Monthly Magazine. Edited by John Clarke Redpath and Helen H. Gardner. Price, $3.00 per year, in advance. Single Number, 25 Cents. Boston: The Arena Co.

The April number of *The Arena* is fully up to the average. In the opening article Governor Pingree, Mayor of Detroit, continues the discussion of Municipal Reform begun in the March number by Mayor Quincy, of Boston. Mayor Pingree, in his breezy paper, affirms that "contracts are the centre and almost the entire circumference of municipal government," and that "almost all the bribes of serious influence in municipalities are given for contracts." His remedy is the letting of contracts by referendum, or direct popular vote.

Under the title of "Lincoln and the Matson Negroes," Jesse W. Weik details the history of a curious slave case, the records of which he has recently unearthed, in which Lincoln was concerned, and which was tried in the circuit court in Illinois, in 1847, during the old fugitive-slave day. None of the numerous biographies of Lincoln makes mention of his part in the affair.

B. F. F.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single Number, 25 cents. The Review of Reviews Co., 13 Astor Place, New York City.

In the "Progress of the World" department of the April *Review of Reviews*, the editor comments on the change of administration at Washington, on the tariff bill, and other measures before the extra session of Congress, and on President McKinley's diplomatic appointments; the Greco-Cretan situation is carefully reviewed, and other recent developments in foreign politics are treated with the thoroughness and impartiality to which the *Review's* readers have grown accustomed.

B. F. F.

GUILLAUME JULES HOÜEL

THE
AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. IV. MAY, 1897. No. 4.

DISCUSSION OF MERIT CONTESTS IN COLLEGE EXAMINATIONS BY THE METHOD OF LEAST SQUARES.

By CHARLES H. KUMMELL, U. S. Coast and Geodetic Survey, Washington, D. C.

[A paper read before the Philosophical Society of Washington.]

It is well known that it is customary in schools and colleges to estimate merit on the basis of 100 being perfect. If then a number of students 1, 2, 3, ...n receive from judges A, B, C, ...L (number$=m$) the estimates of merit a_1, b_1, c_1, ... l_1 ; a_2, b_2, c_2, ...l_2 ; ...... respectively, it is required to find from these discrepant data the most probable merit of each student as well as the personal error of each judge.

Take the case in which three judges A, B, C, have given estimates of merit to seven students, and let the estimates for the first student be:

$a_1=87$; $b_1=70$; $c_1=70$.

Let $M_1=$true or most probable merit.

$\Delta a_1=$error of judge A,

$\Delta b_1=$error of judge B,

$\Delta c_1=$error of judge C,

then we have undoubtedly,

$$M_1=\tfrac{1}{3}(a_1+b_1+c_1)\pm0.6745\sqrt{\frac{\Delta a_1^{\,2}+\Delta b_1^{\,2}+\Delta c_1^{\,2}}{3\times2}}$$

$$=75.7\pm3.82 \text{ with weight } p_1=1.56.$$

If there was but this one student, then this would be the final answer to the question and the personal errors of the judges must be taken,

$\Delta a_1 = -11.3,$

$\Delta b_1 = +5.7,$

$\Delta c_1 = +5.7.$

If these same judges give estimates of merit to more than one student, we shall have more or less discrepant values of the errors of the judges from which a mean personal error may be determined. Now we have the following individual results for the seven students in the example,

$M_1 = 87 - 11.3 = 70 + 5.7 = 70 + 5.7 = 75.7 \pm 3.82$; $p_1 = 1.56,$

$M_2 = 92 - 4.3 = 76 + 11.7 = 95 - 7.3 = 87.7 \pm 3.98$; $p_2 = 1.44,$

$M_3 = 80 - 11.7 = 60 + 8.3 = 65 + 3.3 = 68.3 \pm 4.05$; $p_3 = 1.38,$

$M_4 = 93 - 8.3 = 68 + 16.7 = 93 - 8.3 = 84.7 \pm 5.62$; $p_4 = 0.71,$

$M_5 = 85 - 11.7 = 67 + 6.3 = 68 + 5.3 = 73.3 \pm 3.94$; $p_5 = 1.47,$

$M_6 = 95 - 7.3 = 78 + 9.7 = 90 - 2.3 = 87.7 \pm 3.40$; $p_6 = 1.98,$

$M_7 = 96 - 9.0 = 80 + 7.0 = 85 + 2.0 = 87.0 \pm 3.19$; $p_7 = 2.24.$

In examining these results we notice that judge A always over-estimates, judge B under-estimates, and that judge C is the least consistent of the three judges, as can be roughly seen from their ranges, being 7.4 for A, 11.0 for B, and 14.0 for C. To determine the personal errors of the judges, the arithmetical mean of their errors might be taken, but it is more rigorous to take their weighted mean, using the above weights, which are reciprocally proportional to the squares of the probable errors. We have thus,

$$\Delta a = \frac{[p\Delta a]}{[p]} \pm 0.6745 \sqrt{\frac{[p\Delta a^2] - [p]\Delta a^2}{[p](n-1)}}$$

=personal error of judge A, and similarly for the other judges. We then have the numerical results :

$\Delta a = -9.0 \pm 0.70$; $P_a = 1.53.$

$\Delta b = +8.7 \pm 0.79$; $P_b = 1.20.$

$\Delta c = +0.4 \pm 1.31$; $P_c = 0.44.$

It is obvious that if we correct the original estimates by these quantities, the corrected merits by judge A will be nearest the truth, those of B will be next best, and those of C will hardly be improved ; hence the merits must be redetermined by the formula,

$$M_r = a_r + \Delta a + \Delta^2 a_r = b_r + \Delta b + \Delta^2 b_r = c_r + \Delta c + \Delta^2 c_r,$$

$$= \frac{P_a(a_r + \Delta a) + P_b(b_r + \Delta b) + P_c(c_r + \Delta c)}{P_a + P_b + P_c}$$

$$\pm 0.6745 \sqrt{\frac{P_a(\Delta a^2 + \Delta^2 a_r^2) + P_b(\Delta b^2 + \Delta^2 b_r^2) + P_c(\Delta c^2 + \Delta^2 c_r^2)}{(P_a + P_b + P_c)(3-1)}}$$

which gives the following numerical results :

$M_1 = 78.0 - 0.8 = 78.7 - 1.5 = 70.4 + 6.8 = 77.2 \pm 4.14,$

$M_2 = 83.0 + 2.4 = 84.7 + 0.7 = 95.4 - 10.0 = 85.4 \pm 4.33,$

$M_3 = 71.0 - 1.7 = 68.7 + 0.6 = 65.4 + 3.9 = 69.3 \pm 4.03,$

$M_4 = 84.0 - 1.5 = 76.7 + 5.8 = 93.4 - 10.9 = 82.5 \pm 4.72,$

$M_5 = 76.0 - 1.2 = 75.7 - 0.9 = 68.4 + 6.4 = 74.8 \pm 4.12,$

$M_6 = 86.0 + 0.9 = 86.7 + 0.2 = 90.4 - 3.5 = 86.9 \pm 3.98,$

$M_7 = 87.0 + 0.4 = 88.7 - 1.3 = 85.4 + 2.0 = 87.4 \pm 3.97.$

These results show, as they should, that the corrected estimates of judge *A* came nearest to the correct value, *B's* next best, and *C's* hardly improved ; their ranges being 4.1, 7.3 and 17.7 respectively. We also notice that now contestant 7 has the highest merit while in the first approximation 2 and 6 came out best with a tie. The reason for this is that 2 and 6 received very high marks from judge *C*, which have very small weight in the second approximation. There is an apparent paradox in this, that the best values of the merits have nevertheless larger probable errors than those of the first approximation. Now each of the arithmetic means of the first approximation involves only three errors out of the 27. By correcting the estimates by the personal errors of the judges, a secondary effect of the remaining 24 errors is added in each case.

The second approximation, although sufficiently close to the true values may however, yet be improved. For we now have more correct actual errors of the judges as follows :

$\Delta a_1 = -9.8$; $\Delta b_1 = +7.2$; $\Delta c_1 = +7.2,$

$\Delta a_2 = -6.6$; $\Delta b_2 = +9.4$; $\Delta c_2 = -9.6,$

$\Delta a_3 = -10.7$; $\Delta b_3 = +9.3$; $\Delta c_3 = +4.3,$

$\Delta a_4 = -10.5$; $\Delta b_4 = +14.5$; $\Delta c_4 = -10.5,$

$\Delta a_5 = -10.2$; $\Delta b_5 = +7.8$; $\Delta c_5 = +6.8,$

$\Delta a_6 = -8.1$; $\Delta b_6 = +8.9$; $\Delta c_6 = -3.1,$

$\Delta a_7 = -8.6$; $\Delta b_7 = +7.4$; $\Delta c_7 = +2.4.$

The weighted means of these, weights being given according to their probable errors, will be new values of the personal errors Δa, Δb, Δc of the judges and applying these we obtain new values for the merits. It is easily seen however that this can only affect the 0.01 and though theoretically speaking an infinite number of approximations is required to obtain the most probable values of the merits we may safely regard the second approximation as sufficient.

ON THE CIRCULAR POINTS AT INFINITY.

By E. D. ROE, JR., Associate Professor of Mathematics in Oberlin College, Oberlin, Ohio.

I. The Coordinate System. In the following discussion, in addition to the usual Cartesion coördinates, homogeneous point and line coördinates will be used. They are related to Cartesian coördinates as follows :*

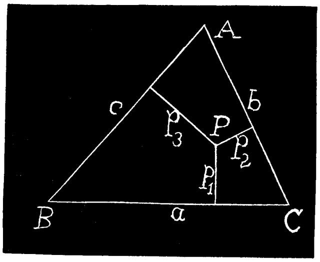


Fig. 1.

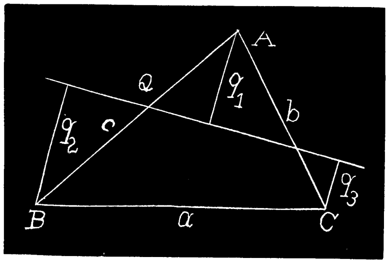


Fig. 1'.

In figure 1, p_1, p_2, p_3 are the perpendicular distances of a point P, from the sides of the coördinate triangle.

In figure 1', q_1, q_2, q_3 are the perpendicular distances of a line Q, from the vertices of the coördinate triangle.

The three point coördinates of P are expressed as follows :

$$\begin{aligned} \rho x_1 &= p_1 \varkappa_1 \\ \rho x_2 &= p_2 \varkappa_2 \\ \rho x_3 &= p_3 \varkappa_3 \end{aligned} \qquad (1)$$

The three line coördinates of Q are expressed as follows :

$$\begin{aligned} \sigma u_1 &= q_1 \lambda_1 \\ \sigma u_2 &= q_2 \lambda_2 \\ \sigma u_3 &= q_3 \lambda_3 \end{aligned} \qquad (1)'$$

The $\varkappa$'s and λ's are six constants which might be chosen at pleasure, but for convenience are chosen in a particular way.

*For a fuller treatment see Clebsch, Vorlesungen ueber Geometrie, S. 27-29, S 62-78.

In Cartesian

point coördinates the equations of the sides of the triangle may be

$$\begin{aligned} a_1x+b_1y+c_1&=0 \text{ for } BC\\ a_2x+b_2y+c_2&=0 \text{ for } CA\\ a_3x+b_3y+c_3&=0 \text{ for } AB \end{aligned} \qquad (2)$$

line coördinates the equations of the opposite vertices will be

$$\begin{aligned} A_1u+B_1v+C_1&=0 \text{ for } A\\ A_2u+B_2v+C_2&=0 \text{ for } B\\ A_3u+B_3v+C_3&=0 \text{ for } C \end{aligned} \qquad (2)'$$

We have then

$$p_1=\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}, \quad p_2=\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}, \quad p_3=\frac{a_3x+b_3y+c_3}{\sqrt{a_3^2+b_3^2}} \qquad (3)$$

$$q_1=\frac{A_1u+B_1v+C_1}{C_1\sqrt{u^2+v^2}}, \quad q_2=\frac{A_2u+B_2v+C_2}{C_2\sqrt{u^2+v^2}}, \quad q_3=\frac{A_3u+B_3v+C_3}{C_3\sqrt{u^2+v^2}} \qquad (3)'$$

We choose

$$\varkappa_1=\sqrt{a_1^2+b_1^2}, \quad \varkappa_2=\sqrt{a_2^2+b_2^2}, \quad \varkappa_3=\sqrt{a_3^2+b_3^2}$$

$$\lambda_1=C_1, \quad \lambda_2=C_2, \quad \lambda_3=C_3$$

and write

$$\begin{aligned} \rho x_1&=a_1x+b_1y+c_1\\ \rho x_2&=a_2x+b_2y+c_2\\ \rho x_3&=a_3x+b_3y+c_3 \end{aligned} \qquad (A)$$

$$\begin{aligned} \sigma u_1&=A_1u+B_1v+C_1\\ \sigma u_2&=A_2u+B_2v+C_2\\ \sigma u_3&=A_3u+B_3v+C_3 \end{aligned} \qquad (A)'$$

Solving these equations, and writing r for (abc),

$$x=\frac{\rho(A_1x_1+A_2x_2+A_3x_3)}{r}, \quad y=\frac{\rho(B_1x_1+B_2x_2+B_3x_3)}{r}, \quad 1=\frac{\rho(C_1x_1+C_2x_2+C_3x_3}{r}$$

$$u=\frac{\sigma r(a_1u_1+a_2u_2+a_3u_3}{r^2}, \quad v=\frac{\sigma r(b_1u_1+b_2u_2+b_3u_3}{r^2}, \quad 1=\frac{\sigma r(c_1u_1+c_2u_2+c_3u_3}{r^2}$$

or

$$x=\frac{A_1x_1+A_2x_2+A_3x_3}{C_1x_1+C_2x_2+C_3x_3}, \quad y=\frac{B_1x_1+B_2x_2+B_3x_3}{C_1x_1+C_2x_2+C_3x_3} \qquad (5)$$

$$u=\frac{a_1u_1+a_2u_2+a_3u_3}{c_1u_1+c_2u_2+c_3u_3}, \quad v=\frac{b_1u_1+b_2u_2+b_3u_3}{c_1u_1+c_2u_2+c_3u_3} \qquad (5)'$$

Upon our choice of the $\varkappa$'s and λ's depends the following important result :

$$\rho\sigma(u_1x_1+u_2x_2+u_3x_3)=r(ux+vy+1). \tag{6}$$

Hence $u_1x_1+u_2x_2+u_3x_3$ vanisnes whenever $ux+vy+1$ vanishes. We may note that

(7)

$C_1x_1+C_2x_2+C_3x_3=0$ is the equation of the line at infinity. It gives the condition that $x=y=\infty$.

(7)′

$c_1u_1+c_2u_2+c_3u_3=0$, is the equation of the origin of coördinates. It gives the condition that $u=v=\infty$.

II. The Circular Points.*

A. Proof that all circles whatsoever pass through two points at infinity.

The equations of any two circles may be written,

$$\begin{aligned} x^2+y^2+2gx+2fy+c=0.\\ x^2+y^2+2g'x+2f'y+c'=0.\end{aligned} \tag{8}$$

In homogeneous point coördinates,

$$x=\frac{A_1x_1+A_2x_2+A_3x_3}{C_1x_1+C_2x_2+C_3x_3}=\frac{A}{C}$$

by (5) and our equations become,

$$y=\frac{B_1x_1+B_2x_2+B_3x_3}{C_1x_1+C_2x_2+C_3x_3}=\frac{B}{C}.$$

$$\begin{aligned} A^2+B^2+2gAC+2fBC+cC^2=0.\\ A^2+B^2+2g'AC+2f'BC+c'C^2=0.\end{aligned} \tag{9}$$

The lines passing through their points of intersection are, by subtraction,

$$C[2(g-g')A+2(f-f')B+(c-c')C]=0. \tag{10}$$

Of these the line $C=0$, is the one that interests us. It is the equation of the line at infinity. The infinite points are found by solving $C=0$, with the equation of either circle, and thus we find them from $C=0$, and $A^2+B^2=0$, or in Cartesian coördinates from the equation of the line at infinity and $x^2+y^2=0$; this would give the same solution always for any two circles ; therefore every circle passes through two points at infinity.

B. Cartesian equation of the points in line coördinates.

The equations of the points in Cartesian line coördinates may be readily obtained.

*See Clebsch, Vorlesungen ueber Geometrie, S. 145-149. Salmon's Conic Sections, pages 238, 325. Fiedler's Salmon, Analytische Geometrie der Kegelschnitte, S. 208.

As $x = \frac{A}{C}$, $y = \frac{B}{C}$, the equation of a point $ux + vy + 1 = 0$, becomes

$$Au + Bv + C = 0.$$

$x + yi = 0$, gives for any point on the line $\quad A + Bi = 0.$
We must then have for one of the points $Au + Bv + C = 0$, all true.

$$A + Bi = 0$$
$$C = 0$$

$$\text{Hence} \begin{vmatrix} u & v & 1 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0, \text{ or } ui - v = 0,$$

similarly for the other point $-ui - v = 0$.

For the pair we have $u^2 + v^2 = 0$. (11)

C. Coördinates of the circular points.

1. *Homogeneous rectangular coördinates.* We saw that we could find the coördinates of the circular points by solving the equations of the line at infinity, $x^2 + y^2 = 0$. The equation of the line at infinity is $0x + 0y + c = 0$. $x^2 + y^2 = (x + yi)(x - yi) = 0$, a pair of imaginary straight lines through the origin. We will find the intersections of a line $ax + by + c = 0$ with $x + yi = 0$, and $x - yi = 0$.

Solving $ax + by = -c$, we get $x = \dfrac{-ci}{ai - b}$.

$$x + yi = 0 \qquad\qquad y = \frac{c}{ai - b}.$$

Or $\dfrac{x}{-i} = \dfrac{y}{1} = \dfrac{c}{ai - b}$. If now we introduce a third coördinate c to make our rectangular coöordinates homogeneous, and consider the ratios of x, y, and c as coördinates, we see that the coördinates of one imaginary circular point are given by $\quad x : y : c = -i : 1 : 0,$
and of the other by $\quad x : y : c = i : 1 : 0. \qquad (12)$

2. *Homogeneous point coöordinates.* The coördinates of the circular points however assume a more convenient form when expressed in the general point coödinates. We shall obtain them in proving the following highly interesting proposition :

A circle, with fixed center in the finite region, whose radius becomes indefinitely great, degenerates into the two circular points at infinity.

We will obtain the equation of the circle in homogeneous line coördinates. We express that a line u is always at a distance r from the point x', the center of the circle.

Let $u_1x_1 + u_2x_2 + u_3x_3 = 0$ be the point equation of the line [see (6)].
This by (4) is

$$u_1(a_1x + b_1y + c_1) + u_2(a_2x + b_2y + c_2) + u_3(a_3x + b_3y + c_3) = 0$$

or $(u_1a_1+u_2a_2+u_3a_3)x+(u_1b_1+u_2b_2+u_3b_3)y+(u_1c_1+u_2c_2+u_3c_3)=0.$ (13)

Putting this in cosine form, and taking the square of the distance from $x'y'$ to the line equal to r^2, we get,

$$\frac{[(u_1a_1+u_2a_2+u_3a_3)x'+(u_1b_1+u_2b_2+u_3b_3)y'+(u_1c_1+u_2c_2+u_3c_3)]}{(u_1a_1+u_2a_2+u_3a_3)^2+(u_1b_1+u_2b_2+u_3b_3)^2}=r^2.$$

or (14)*

$$\frac{(u_1x_1'+u_2x_2'+u_3x_3')^2}{\varkappa_1^2u_1^2+\varkappa_2^2u_2^2+\varkappa_3^2u_3^2-2\varkappa_1\varkappa_2u_1u_2\cos C-2\varkappa_2\varkappa_3u_2u_3\cos A-2\varkappa_3\varkappa_1u_3u_1\cos B}=r^2$$

The reductions in the denominator depend on the following :

$$a_1^2+b_1^2=\varkappa_1^2,\text{ etc.}\quad a_1a_2+b_1b_2=\varkappa_1\varkappa_2\left(\frac{a_1b_2}{\varkappa_1\varkappa_2}+\frac{b_1b_2}{\varkappa_1\varkappa_2}\right)$$

$$=\varkappa_1\varkappa_2(\cos\alpha_1\cos\alpha_2-\sin\alpha_1\sin\alpha_2)=\varkappa_1\varkappa_2\cos C,\text{ etc.},$$

where α_1, α_2, α_3 are the angles that p_1, p_2, p_3 make with the axis of x, and the origin is taken within the coördinate triangle. (14) is the line equation of the circle. We notice now that the expression in the denominator may be factored, for considering the variables as, $\varkappa_1u_1$, $\varkappa_2u_2$, $\varkappa_3u_3$, its discriminant is

$$\begin{vmatrix} 1 & -\cos C & -\cos B \\ -\cos C & 1 & -\cos A \\ -\cos B & -\cos A & 1 \end{vmatrix}$$

and that this is zero, may be shown as follows :

From trigonometry we have the three equations :

$$\begin{aligned} a-b\cos C-c\cos B&=0 \\ -a\cos C+b-c\cos A&=0 \\ -a\cos B-b\cos A+C&=0 \end{aligned}$$

whence it follows that the determinant of the coefficients of a, b, and c, vanishes. Put $\varkappa_1u_1=l$, etc. Then

$$l^2+m^2+n^2-2lm\cos C-2mn\cos A-2nl\cos B=(l\alpha+m\beta+n\gamma)(l\alpha'+m\beta'+n\gamma')$$

$$=l^2\alpha\alpha'+m^2\beta\beta'+n^2\gamma\gamma'+lm(\alpha\beta'+\alpha'\beta)+mn(\beta\gamma'+\beta'\gamma)+nl(\alpha\gamma'+\alpha'\gamma),$$

and we must have :

*Compare Salmon's Conic Sections, page 128, Ex. 6.

$$\begin{array}{lll} \alpha\alpha'=1. & \text{Take } \alpha=\cos B-i\sin B, & \gamma=-1, \\ \beta\beta'=1. & \alpha'=\cos B+i\sin B, & \gamma'=-1, \\ \gamma\gamma'=1. & \beta=\cos A+i\sin A, & \\ & \beta'=\cos A-i\sin A, & \end{array}$$

and the first three conditions are satisfied. Also,

$$\alpha\beta'+\alpha'\beta=[\cos(A+B)-i\sin(A+B)+\cos(A+B)+i\sin(A+B)]$$
$$=2\cos(A+B)=-2\cos C.$$
$$\beta\gamma'+\beta'\gamma=-\cos A-i\sin A-\cos A+i\sin A=-2\cos A.$$
$$\alpha\gamma'+\alpha'\gamma=-\cos B+i\sin B-\cos B-i\sin B=-2\cos B.$$

Our expression therefore has the two factors, viz :

$$[(\cos B-i\sin B)\varkappa_1 u_1+(\cos A+i\sin A)\varkappa_2 u_2-\varkappa_3 u_3]=L.$$

$$[(\cos B+i\sin B)\varkappa_1 u_1+(\cos A-i\sin A)\varkappa_2 u_2-\varkappa_3 u_3]=M.$$

Also write $u_1x_1+u_2x_2+u_3x_3=u_x$, then our line equation of the circle may be written

$$LM=\left(\frac{u_{x'}}{r}\right)^2. \qquad (15)$$

If γ becomes indefinitely great $u_{x'}$ does not become indefinitely great, for the x''s are finite, the coördinates of the fixed center, and the u's by (4)$'$ are always finite, since u and v are always finite, and for a line which is moved off to infinity approach zero together. It follows that $\lim\limits_{r\doteq\infty}\left(\frac{u_{x'}}{r}\right)=0.$

Hence the equation of a circle whose radius is infinite, and whose center is in the finite region is in line coördinates,

$$LM=0. \qquad (16)$$

But this is also the equation of a point-pair, and since we have proved that every circle whatsoever contains the two imaginary circular points at infinity, it follows that the two points into which this circle has degenerated are themselves the two imaginary circular points at infinity. As we might just as well have factored our expression LM in two other ways, in which the two angles B and C, or C and A, play the same part as A and B, we may write the coördinates of the two points in the three ways, as follows:

$$\begin{array}{llll} (1). & -\varkappa_1, & \varkappa_2 e^{-iC}, & \varkappa_3 e^{iB}. \\ & -\varkappa_1, & \varkappa_2 e^{iC}, & \varkappa_3 e^{-iB}. \\ (2). & \varkappa_1 e^{iC}, & -\varkappa_2, & \varkappa_3 e^{-iA}. \\ & \varkappa_1 e^{-iC}, & -\varkappa_2, & \varkappa_3 e^{iA}. \\ (3). & \varkappa_1 e^{-iB}, & \varkappa_2 e^{iA}, & -\varkappa_3. \\ & \varkappa_1 e^{iB}, & \varkappa_2 e^{-iA}, & -\varkappa_3. \end{array} \qquad (17).$$

Before going on to the next division of our discussion, we will recall* that if

$$a_{11}x_1^2+a_{22}x_2^2+a_{33}x_3^2+2a_{12}x_1x_2+2a_{23}x_2x_3+2a_{31}x_3x_1=0$$

where $a_{i\kappa}=a_{\kappa i}$ is the point equation of a conic, with non-vanishing discriminant,

$$A_{11}u_1^2+A_{22}u_2^2+A_{33}u_3^2+2A_{12}u_1u_2+2A_{23}u_2u_3+2A_{31}u_3u_1=0$$

where $A_{i\kappa}=A_{\kappa i}$ is the line equation of a conic, with non-vanishing discriminant,

then

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & u_1 \\ a_{21} & a_{22} & a_{23} & u_2 \\ a_{31} & a_{32} & a_{33} & u_3 \\ u_1 & u_2 & u_3 & 0 \end{vmatrix}=0. \qquad (18)$$

is the line equation of the same conic.

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} & x_1 \\ A_{21} & A_{22} & A_{23} & x_2 \\ A_{31} & A_{32} & A_{33} & x_3 \\ x_1 & x_2 & x_3 & 0 \end{vmatrix}=0. \qquad (18)'$$

is the point equation of the same conic.

and that always

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & x_1 & x_1' \\ a_{21} & a_{22} & a_{23} & x_2 & x_2' \\ a_{31} & a_{32} & a_{33} & x_3 & x_3' \\ x_1 & x_2 & x_3 & 0 & 0 \\ x_1' & x_2' & x_3' & 0 & 0 \end{vmatrix}=0 \qquad (19)$$

is the equation of the pair of tangents from the point x' to the same conic.

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} & u_1 & u_1' \\ A_{21} & A_{22} & A_{23} & u_2 & u_2' \\ A_{31} & A_{32} & A_{33} & u_3 & u_3' \\ u_1 & u_2 & u_3 & 0 & 0 \\ u_1' & u_2' & u_3' & 0 & 0 \end{vmatrix}=0 \qquad (19)'$$

is the equation of the pair of points where the line u' cuts the same conic.

III. Fundamental Geometrical Relations Defined in Terms of the Circular Points at Infinity.

A. The equation of a circle in terms of the circular points.

Our line equation of the circle (14) may be written :

$$(r^2\varkappa_1^2-x_1'^2)u_1^2+(r^2\varkappa_2^2-x_2'^2)u_2^2+(r^2\varkappa_3^2-x_3'^2)u_3^2$$
$$-2(r^2\varkappa_1\varkappa_2\cos B+x_1'x_2')u_1u_2-2(r^2\varkappa_2\varkappa_3\cos A+x_2'x_3')u_2u_3$$
$$-2(r^2\varkappa_3\varkappa_1\cos B+x_3'x_1')u_3u_1=0.$$

Therefore by (18)', its point equation is,

$$\begin{vmatrix} r^2\varkappa_1^2-x_1'^2 & -(r^2\varkappa_1\varkappa_2\cos C+x_1'x_2') & -(r^2\varkappa_2\varkappa_3\cos B+x_1'x_3') & x_1 \\ -(r^2\varkappa_1\varkappa_2\cos C+x_1'x_2') & r^2\varkappa_2^2-x_1'^2 & -(r^2\varkappa_2\varkappa_3\cos A+x_2'x_3') & x_2 \\ -(r^2\varkappa_1\varkappa_3\cos B+x_1'x_3') & -(r^2\varkappa_2\varkappa_3\cos A+x_2'x_3') & r^2\varkappa_3^2-x_3' & x_3 \\ x_1 & x_2 & x_3 & 0 \end{vmatrix}=0.$$

The coefficients of x_1^2 and x_1x_2 changed in sign will be :

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*Clebsch, Vorlesungen ueber Geometrie, S. 113.

Of x_1^2, $(r^2\varkappa_2^2-x_2'^2)(r^2\varkappa_3^2-x_3'^2)-(r^2\varkappa_2\varkappa_3\cos A+x_2'x_3')^2$

$$=r^4\varkappa_2^2\varkappa_3^2-r^2(\varkappa_3^2x_2'^2+\varkappa_2^2x_3'^2)-r^4\varkappa_2^2\varkappa_3^2\cos^2A-2r^2\varkappa_2\varkappa_3x_2'x_3'\cos A,$$

of x_1x_2, $2(r^2\varkappa_3^2-x_3'^2)(r^2\varkappa_1\varkappa_2\cos C+x_1'x_2')$

$$+2(r^2\varkappa_2\varkappa_3\cos A+x_2'x_3')(r^2\varkappa_1\varkappa_3\cos B+x_1'x_3')$$

$$=2(r^4\varkappa_1\varkappa_2x_3^2\cos C-r^2\varkappa_1\varkappa_2\cos Cx_3'^2+r^2\varkappa_3^2x_1'x_2'+r^4\varkappa_1\varkappa_2\varkappa_3\cos A\cos B$$
$$+r^2\varkappa_2\varkappa_3x_1'x_3'\cos A+r^2\varkappa_1\varkappa_3x_2'x_3'\cos B).$$

Similarly for the other terms. Put terms containing r^2 on right hand side of equation, divide by r^2, arrange, and reduce, and we get finally,

$$r^2(\varkappa_2\varkappa_3\sin Ax_1+\varkappa_3\varkappa_1\sin Bx_2+\varkappa_1\varkappa_2\sin Cx_3)^2$$
$$=\varkappa_3^2(x_2'x_1-x_2x_1')^2+\varkappa_1^2(x_3'x_2-x_3x_2')^2+\varkappa_2^2(x_1'x_3-x_1x_3')^2$$
$$-2\varkappa_2\varkappa_3(x_3x_1'-x_3'x_1)(x_1x_2'-x_1'x_2)\cos A-2\varkappa_3\varkappa_1(x_1x_2'-x_1'x_2)(x_2x_3'-x_2'x_3)\cos B$$
$$-2\varkappa_1\varkappa_2(x_2x_3'-x_2'x_3)(x_3x_1'-x_3x_1)\cos C.^* \qquad (20)$$

From what we did with LM of (14) it is clear that the right member of (20) can be factored. Put $(x_3'x_2-x_3x_2')=l$, etc. We then have as factors,

$$\begin{array}{ll} l\varkappa_1e^{-iB}+m\varkappa_2e^{iA}-n\varkappa_3 \text{ or if} & \rho\xi_1=\varkappa_1e^{-iB},\ \rho\xi_1'=\varkappa_1e^{iB}. \\ l\varkappa_1e^{iB}+m\varkappa_2e^{-iA}-n\varkappa_3 & \rho\xi_2=\varkappa_2e^{iA},\ \rho\xi_2'=\varkappa_2e^{-iA}. \\ & \rho\xi_3=-\varkappa_3,\ \rho\xi_3'=-\varkappa_3. \end{array}$$

The factors become,

$$\rho(l\xi_1+m\xi_2+n\xi_3)$$
$$\rho(l\xi_1'+m\xi_2'+n\xi_3')$$

and by supplying the values of l, m, and n, these become

$$\rho\begin{vmatrix}\xi_1 & x_1 & x_1'\\ \xi_2 & x_2 & x_2'\\ \xi_3 & x_3 & x_3'\end{vmatrix}, \qquad \rho\begin{vmatrix}\xi_1' & x_1 & x_1'\\ \xi_2' & x_2 & x_2'\\ \xi_3' & x_3 & x_3'\end{vmatrix}$$

Also the left member is $r^2(C_1x_1+C_2x_2+C_3x_3)^2$ since

$$\varkappa_2\varkappa_3\sin A=\varkappa_2\varkappa_3\sin(\alpha_3-\alpha_2)\dot{=}\varkappa_2\varkappa_3\left(\frac{a_2b_3-a_3b_2}{\varkappa_2\varkappa_3}\right)=C_1,\text{ etc.},$$

*Compare Salmon's Conic Sections, page 128, Ex. 6.

and (20) takes the form

$$r^2(C_1x_1+C_2x_2+C_3x_3)^2=\rho^2(\xi xx')(\xi' xx'). \qquad (21)$$

As a check on our determination of the coördinates of the circular points at infinity, let us see if the coördinates, $\rho\xi$, $\rho\xi'$ satisfy this equation of the circle. They certainly reduce the right member to zero, for $(\xi\xi x')=0$, and $(\xi'\xi' x')=0$. They also reduce the left member to zero, for it is reduced to zero, by the coördinates of any infinitely distant points [see (7)]. We thus confirm the results of (17), and (21) is the equation of a circle, whose center is at x', in terms of the coördinates of the circular points at infinity.*

B. General distance formula in terms of the circular coördinates.

The preceding result (21) may be used as a formula for the distance between two points x and x'. It must first however be made homogeneous in all the coördinates. It is clear that,

$$C_1x_1+C_2x_2+C_3x_3=\kappa\rho^2\begin{vmatrix} x_1 & \xi_1 & \xi_1' \\ x_2 & \xi_2 & \xi_2' \\ x_3 & \xi_3 & \xi_3' \end{vmatrix},$$

for the vanishing of both expressions signifies a line through t
tant points. To determine κ, let $x_2=x_3=0$.

$$\text{Then } C_1x_1=\kappa\rho^2\begin{vmatrix} x_1 & \xi_1 & \xi_1' \\ 0 & \xi_2 & \xi_2' \\ 0 & \xi_3 & \xi_3' \end{vmatrix}$$

$$=\kappa\rho^2x_1\frac{(-\kappa_2\kappa_3(\cos A+i\sin A)+\kappa_2\kappa_3(\cos A-i\sin A)}{\rho^2}$$

$$=-2\kappa x_1\kappa_2\kappa_3 i\sin A$$

$$=-2\kappa x_1 iC_1.$$

$$\therefore\ \kappa=-\frac{1}{2i}=\tfrac{1}{2}i, \text{ and we have}$$

$$C_1x_1+C_2x_2+C_3x_3=\tfrac{1}{2}(i\rho^2)(x\xi\xi')=$$

a constant by the first solution of (4).

By this fact we can make r^2 homogeneous. For

$$r^2=c\frac{(\xi xx')(\xi' xx')}{(x\xi\xi')^2(x'\xi\xi)^2}$$

where c is some constant. To determine it let the distance between a and a' be

*A question might arise as to the constant ρ^2. That can be disposed of as is done in the next division.

unity. Substituting the value of c so obtained we have

$$r^2 = \frac{(\xi x x')(\xi' x x')(a \xi \xi')^2 (a' \xi \xi')^2}{(x \xi \xi')^2 (x' \xi \xi')^2 (\xi a a')(\xi' a a')}, \qquad (22)$$

which is seen to be homogeneous and of degree zero in all the coördinates. It is also clear that it is an absolute invariant expression in ternary forms, for on account of the multiplication law of determinants any determinant of the form $(x'y'z')$ in terms of the new variables becomes equal under linear transformation, to $M(xyz)$ where M is the modulus of the transformation, and the transformation equations are:

$$\begin{aligned}
\rho x_1' &= a_{11} x_1 + a_{12} x_2 + a_{13} x_3 \\
\rho x_2' &= a_{21} x_1 + a_{22} x_2 + a_{23} x_3 \\
\rho x_3' &= a_{31} x_1 + a_{32} x_2 + a_{33} x_3 \quad *
\end{aligned}$$

Our distance is thus defined projectively with respect to the circular points.

C. The equation of the pair of lines from a point x to the two circular points.

These lines will certainly be given by $(xx'\xi)(xx'\xi') = 0$. (23).

By (16) the equation of the circular points in line coördinates is

$$\varkappa_1^2 u_1^2 + \varkappa_2 u_2^2 + \varkappa_3 u_3^2 - 2\varkappa_1 \varkappa_2 u_1 u_2 \cos C - 2\varkappa_2 \varkappa_3 u_2 u_3 \cos A - 2\varkappa_3 \varkappa_1 u_3 u_1 \cos B = 0.$$

Now the equation of the pair of tangents from x', to the points will be by (19),

$$\begin{vmatrix}
\varkappa_1^2 & -\varkappa_1 \varkappa_2 \cos C & -\varkappa_1 \varkappa_3 \cos B & x_1 & x_1' \\
-\varkappa_1 \varkappa_2 \cos C & \varkappa_2^2 & -\varkappa_2 \varkappa_3 \cos A & x_2 & x_2' \\
-\varkappa_1 \varkappa_3 \cos B & -\varkappa_2 \varkappa_3 \cos A & \varkappa_3^2 & x_3 & x_3' \\
x_1 & x_2 & x_3 & 0 & 0 \\
x_1' & x_2' & x_3' & 0 & 0
\end{vmatrix} = 0. \qquad (24)$$

It follows that this determinant is equal to the left member of (23) multiplied by a constant, since their vanishing represents the same geometrical form. Let $\varkappa$ denote this constant. Put $x_1 = a_4$, $x_2' = b_5$, above. $\varkappa_3^2 = c_3$, $x_1 = d_1$, $x_2' = e_2$, below. On the left hand the term containing $x_1^2 x_2'^2$ will be represented by $a_4 b_5 c_3 d_1 e_2$. The number of inversions of order $j = 3 + 3 + 2 = 8$. On the right hand $\varkappa x_1^2 \begin{vmatrix} x_2' & \\ & \xi_3 \end{vmatrix} \begin{vmatrix} x_2' & \\ & \xi_3' \end{vmatrix} = \dfrac{\varkappa x_1^2 x_2' \varkappa_3^2}{\rho^2}$.

*The result of this division is found in Klein's First Lecture, Winter Semester, 1889-90, on the "Nicht-Euclidsche Geometry," S. 40.

$$\therefore\ x_1^2 x_2'^2 \varkappa_3^2 = \frac{\varkappa}{\rho^2} x_1^2 x_2'^2 \varkappa_3^2. \quad \therefore\ \varkappa = \rho^2,$$

and we obtain the interesting result in determinants,*

$$\begin{vmatrix} \varkappa_1^2 & -\varkappa_1\varkappa_2\cos C & -\varkappa_1\varkappa_3\cos B & x_1 & x_1' \\ -\varkappa_1\varkappa_2\cos C & \varkappa_2^2 & -\varkappa_2\varkappa_3\cos A & x_2 & x_2' \\ -\varkappa_1\varkappa_3\cos B & -\varkappa_2\varkappa_3\cos A & \varkappa_3^2 & x_3 & x_3' \\ x_1 & x_2 & x_3 & 0 & 0 \\ x_1' & x_2' & x_3' & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 & x_1' & \varkappa_1 e^{-iB} \\ x_2 & x_2' & \varkappa_2 e^{iA} \\ x_3 & x_3' & -\varkappa_3 \end{vmatrix} \cdot \begin{vmatrix} x_1 & x_1' & \varkappa_1 e^{iB} \\ x_2 & x_2' & \varkappa_2 e^{-iB} \\ x_3 & x_3' & -\varkappa_3 \end{vmatrix} \qquad (25)$$

D. The angle between the lines.

Let us take the four lines,

$$x+iy=0. \quad 1. \qquad x-iy=0. \quad 2.$$
$$x+\lambda y=0. \quad 3. \qquad x+\lambda' y=0. \quad 4.$$

The double ratio of these is, taking them in the order named using the ratio,

$$\alpha = \frac{(\mu_1-\mu_3)(\mu_4-\mu_2)}{(\mu_3-\mu_2)(\mu_1-\mu_4)}, \dagger$$

where $\mu_1 = i$, $\mu_2 = -i$, $\mu_3 = \lambda$, $\mu_4 = \lambda'$, we have

$$\frac{(i-\lambda)(\lambda'+i)}{(\lambda+i)(i-\lambda')} = r + 5i,$$

$$\text{or } i\lambda' - 1 - \lambda\lambda' - i\lambda = ri\lambda - r - r\lambda\lambda' - ri\lambda' - s\lambda - si - s\lambda\lambda' i + s\lambda.$$

Or, equating the real parts, and the imaginary parts, we have,

$$\lambda' - \lambda = r(\lambda - \lambda') - s(1+\lambda\lambda'), \ (\lambda - \lambda')(r+1) = s(1+\lambda\lambda').$$

$$1 + \lambda\lambda' = r(1+\lambda\lambda') + s(\lambda - \lambda'), \ s(\lambda-\lambda') = (1-r)(1+\lambda\lambda').$$

$$\therefore\ \frac{\lambda-\lambda'}{1+\lambda\lambda'} = \frac{s}{1+r} = \frac{1-r}{s}.$$

And we see that r and s are restricted to the relation: $r^2 + s^2 = 1$. If ϕ denote the angle between the lines 3 and 4,

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*Compare Salmon's Conic Sections, page 133, Ex. 2.

†Clebsch, Vorlesungen ueber Geometrie, S. 38.

$$\tan\phi=\frac{s}{1+r}=\pm\sqrt{\frac{1-r}{1+r}}. \quad \therefore\ r=\cos 2\phi,\ s=\pm\sin 2\phi.$$

If we denote our double ratio of the four lines by (DR), and choose the lower sign for s, we have

$$(DR)=\cos 2\phi-i\sin 2\phi, \text{ or}$$
$$(DR)=e^{-2\phi i},$$
$$\phi=\tfrac{1}{2}i\log(DR). \quad (26)$$

We could have obtained this result in another way. If the equations of 3 and 4 were written more generally

$$ux+vy+1=0.$$
$$u'x+v'y+1=0.$$

$\tan\phi=\dfrac{uv'-u'v}{uu'+vv'}$. If $z=x+yi$, we have,

$$\log(x+yi)=\tfrac{1}{2}\log(x^2+y^2)+i\tan^{-1}(y/x).$$

$$\log(x-yi)=\tfrac{1}{2}\log(x^2+y^2)-i\tan^{-1}(y/x).$$

$$\log\frac{x+yi}{x-yi}=2i\tan^{-1}(y/x)=2i\omega, \text{ where } \tan\omega=y/x.$$

$$\omega=\tfrac{1}{2}i\log\frac{x-yi}{x+yi}.$$

Using this as a formula for expressing ϕ, we get

$$\phi=\tfrac{1}{2}i\log\left(\frac{uu'+vv'+i(u'v-uv')}{uu'+vv'-i(u'v-uv')}\right)$$

$$=\tfrac{1}{2}i\log\left(\frac{uu'+vv'+\sqrt{(uu'+vv')^2-(u^2+v^2)(u'^2+v'^2)}}{uu'+vv'-\sqrt{(uu'+vv')^2-(u^2+v^2)(u'^2+v'^2)}}\right).* \qquad (27)$$

But the expression under the logarithm is the quotient of the roots of the equation in λ,

$$u^2+v^2+2\lambda(uu'+vv')+\lambda^2(u'^2+v'^2)=0,†$$

an equation obtained by substituting in the line equation of the circular points, (11), the values $u+\lambda u'$, $v+\lambda v'$, so that the ratio of the two roots of λ is again

*First given by Laguerre: Nouvelles Annales de Math. 1853.
†See Clebsch, Vorlesungen ueber Geometrie, S. 148.

the double ratio of the four lines. Klein in the before mentioned lecture on Non-Euclidean Geometry obtains the same result in still another way. The angle between two lines is thus also defined projectively with reference to the two fixed circular points at infinity, for the double ratio of four lines is an absolute invariant under linear transformation. Some special results in angle determination may interest us.*

1. The angle that a line to either of the two circular points makes with any other line of the finite region is to be regarded as infinite. For the tangent of that angle is given by

$$\frac{\tan \psi - i}{1 + i \tan \psi} = -i,$$

i, and ψ, being the tangents of the two lines. Now we have,

$$\tan x = \frac{1}{i} \tanh xi = \frac{1}{i} \frac{e^{2xi} - 1}{e^{2xi} + 1}. \quad \lim_{x \doteq \infty} \tan x = \frac{1}{i}, \quad (\tan x)_{x=\infty} = -i.$$

The above angle between the two lines must therefore be regarded as an infinite one. Similarly for the line to the other circular point. By our new definition of angle, the matter is simpler still, for then in this case $\lambda = i$, or $-i$, and $(DR) = r + si = 0$, or ∞, whence $\phi = \infty$.

2. Two lines are perpendicular to each other when the double ratio (26) is equal to -1, that is when the four lines form a harmonic quadrupel. For using $-\pi i$ as $\log(-1)$ we get from (26), $\phi = \frac{1}{2}\pi$. Also above, put $r = -1$, $s = 0$, and obtain the same result.

3. Two lines are parallel when $r = 1$, $s = 0$, that is when the double ratio of the four lines is unity.

4. Two lines make an angle of 45°, or 135° when $r = 0$, $s = \pm l$; that is when the double ratio is equal to $\pm i$.

5. All angles inscribed in a circle and intercepting the same arc are equal for the double ratio of four rays from some variable point in a circle to four fixed points in constant. Here the four fixed points are the two finitepoints at the ends of the arc, and the two fixed circular points at infinity. But if the double ratio is constant r and s are constant, therefore,

$$\frac{\lambda - \lambda'}{1 + \lambda\lambda'} = \frac{s}{1 + r} = \frac{1 - r}{s}$$

is constant, and the inscribed angle is constant.

IV. Relation of the Circular Points to Non-Euclidean Geometry.

What we have established in the preceding seems to suggest the way for investigations and generalizations of the greatest importance. And such was the course of history on the analytic side of the passage from Euclidean to Non-Euc-

†Clebsch, Vorlesungen ueber Geometrie, S. 147-149.

lidean geometry. It only remained to make the generalization that, $\Sigma xx = \Sigma a_{i\kappa} x_i x_\kappa = 0$, being the equation of the fundamental form in point or in line coördinates as might be needed, the expression

$$\varkappa \log\left(\frac{\Sigma xx' + \sqrt{(\Sigma xx')^2 - \Sigma xx'.\Sigma x'x}}{\Sigma xx' - \sqrt{\Sigma xx' - \Sigma xx'.\Sigma x'x}}\right)$$

should be in general the distance between the points, or the angle between two lines. If $\varkappa = \frac{1}{2}i$, and $\Sigma xx = u^2 + v^2$ we have the ordinary Euclidean angle between two lines. If $\Sigma xx'$ is not equal to $u^2 + v^2$, we evidently have something quite different from that angle, $\varkappa$ times the logarithm of the double ratio of the two lines and the pair of tangents to the conic from their point of intersection.

The derivation of the Euclidean distance formula is not so simple, a case of limits being involved. According as this fundamental conic is an actual one, a point pair, or an imaginary one, we get hyperbolic, parabolic, or elliptic metrical determination. Cayley seems to have given the first valuable suggestions tending towards analytic methods. Klein has built up an admirable analytic treatment, using what he calls the "Cayley'schen Maassbestimmung" as a basis. In his illustrations of elliptic, and hyperbolic geometry of the plane, he uses as fundamental conics $x^2 + y^2 = -r^2$, and $x^2 + y^2 = r^2$ respectively. It is interesting to note that the square of the element of length in each is,

$$ds_1^2 = \frac{dx^2 + dy^2 + \dfrac{(ydx - xdy)^2}{r^2}}{\left(1 + \dfrac{x^2 + y^2}{r^2}\right)^2}, \qquad ds_3^2 = \frac{dx^2 + dy^2 - \dfrac{(ydx - xdy)^2}{r^2}}{\left(1 - \dfrac{x^2 + y^2}{r^2}\right)^2}.$$

Now if r becomes indefinitely great, we have as the limit of both ds_1^2 and ds_3^2, $ds_2^2 = dx^2 + dy^2$, the square of the element of length in the ordinary Euclidean plane. This affords incidentally confirmation of our proposition under II. C, 2. that when the radius of a circle whose center is in the finite region becomes indefinitely great, the circle degenerates into the two circular points at infinity.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

74. Proposed by JOHN T. FAIRCHILD, Principal of Crawfis College, Crawfis College, Ohio.

When U. S. Bonds are quoted in London at $108\frac{3}{4}$ and in Philadelphia at $112\frac{1}{4}$, exchange \$$4.48\frac{1}{2}$, gold quoted at 107, how much more was a \$1000 U. S. bond worth in London than in Philadelphia ?

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

If I understand the problem correctly, the exchange price is not necessary for the solution.

$\$1000\times1.12\frac{1}{4}=\$1122.50$, price in Philadelphia.

$\$1000\times1.08\frac{3}{4}=\$1087.50$, price in London.

But one dollar of London gold is worth \$1.07 of Philadelphia currency.

$\therefore$ $\$1087.50\times1.07=\$1163.62\frac{1}{2}$, price of London bond in U. S. currency.

$\therefore$ $\$1163.62\frac{1}{2}-\$1122.50=\$41.12\frac{1}{2}$, the amount the London bond cost an American more than the Philadelphia bond.

To find the difference in cost to an Englishman in London, we proceed as follows :

$\$1000\times1.12\frac{1}{4}=\$1122.50$.

$\$1122.53\div1.07=\$1049.06\frac{58}{107}$, price of the Philadelphia bond in English gold.

$\$1000\times1.08\frac{3}{4}=\$1087.50$.

$\$1087.50-\$1049.06\frac{58}{107}=\$38.43\frac{49}{107}$.

$\$38.43\frac{49}{107}\div\$4.89\frac{1}{2}=7£\ 17s\ .433d$.

[We believe Dr. Zerr's view of this problem to be the correct one. EDITOR.]

77. Proposed by F. S. ELDER, Professor of Mathematics, Oklahoma University, Norman, Oklahoma.

For how many seconds must I count the clicking of the rails under a train that the number of rails counted may be equal to the speed of the train in miles per hour, a rail being 30 feet long.

I. Solution by FREDERIC R. HONEY, Ph. B., New Haven, Connecticut, and CHAS. C. CROSS, Laytonsville, Maryland.

This problem is similar to the one proposed in the July-August number, Vol. III. The result is independent of the number of rails counted and of the number of miles per hour the train is running.

In the problem referred to, the answer is $3a/88$ minutes during which the poles are counted, where a equals the number of yards the polls are apart.

In the present case, $a=10$ yards. Hence, substituting, $3a/88$ minutes$=$ $30/88$ minutes$=20\frac{5}{11}$ seconds.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas, and the PROPOSER.

Let t=number of seconds, n=number of miles per hour.

$\therefore$ $5280n/3600=22n/15$ feet per second=speed of train. Also in t seconds train goes $30n$ feet.

$\therefore$ $30n/t$=number of feet in one second.

$\therefore$ $30n/t=22n/15$. $\therefore$ $t=20\frac{5}{11}$ seconds.

78. Proposed by NELSON S. RORAY, South Jersey Institute, Bridgeton, New Jersey.

Solve by pure arithmetic, no algebraic symbols: A Texan farmer owns 5169 cattle; there are 3 times as many horses as cows, plus 569, and 4 times as many cows as sheep, minus 126; how many has he of each? [From *Brooks' Higher Arithmetic*.]

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas, and J. C. CORBIN, Principal of Schools, Pine Bluff, Arkansas.

$5169+126-569=4726$=number of cattle when there are 4 times as many cows as sheep and 3 times as many horses as cows.

Every time he takes 1 sheep, he takes 4 cows and 12 horses, or 17 in all.

$\therefore$ he has as many lots of 1 sheep, 4 cows, 12 horses, as 17 is contained in 4726. $\therefore$ $4726\div17=278$.

$\therefore$ 278×1=number of sheep= 278

$278\times4-126$=number of cows= 986

$278\times12+569$=number of horses=3905

Total=5169

This problem was solved with a different view of its enunciation by Frederic R. Honey, and O. S. Westcott, A. M., Sc. D., Principal North Division High School, Chicago, Illinois.

[NOTE. P. S. Berg and H. C. Wilkes should eaeh have received credit in the last number for solving problems 75 and 76. EDITOR.]

ALGEBRA.

Conducted by J. M, COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

70. Proposed by J. A. CALDERHEAD, A. B., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

Given $\sqrt[3]{(a+x)}+\sqrt[3]{(a-x)}=\sqrt[3]{c}$ to find x.

I. Solution by J. MARCAS BOORMAN, Consultative Mechanician, Counselor at Law, Inventor, Etc., Hewlett, Long Island, New York; EDWARD R. ROBBINS, Master in Mathematics and Physics in Lawrenceville School, Lawrenceville, New Jersey; E. L. SHERWOOD, A. M., Principal of City Schools, West Point, Mississippi; O. W. ANTHONY, M. Sc., Columbian University, Washington, D. C.; A. H. HOLMES, Brunswick, Maine; and J. SCHEFFER, A. M., Hagerstown, Maryland.

Cubing, transposing, etc.,

$$(a^2-x^2)^{\frac{1}{3}}\,[(a+x)^{\frac{1}{3}}+(a-x)^{\frac{1}{3}}]=(c-2a)/3, \text{ or } (a^2-x^2)^{\frac{1}{3}}\,[\,c^{\frac{1}{3}}\,]=(c-2a)/3.$$

$$\therefore\ x=\pm\sqrt{\left(a^2-\frac{(c-2a)^3}{27c}\right)}.\quad \text{If } c=2a,\ x=a.$$

II. Solution by J. H. DRUMMOMD, LL. D., Portland, Maine; A. H. BELL, Hillsboro, Illinois; and CHAS. C. CROSS, Laytonsville, Maryland.

Transposing $\sqrt[3]{(a-x)}$, cubing and reducing, we have,

$$(a-x)^{\frac{2}{3}}-c^{\frac{1}{3}}(a-x)^{\frac{1}{3}}=(2a-c)/3c^{\frac{1}{3}}.$$

Completing the square, we find,

$$(a-x)^{\frac{1}{3}}=\tfrac{1}{2}\left[c^{\frac{1}{3}}\pm\sqrt{\left(\frac{8a-c}{3c^{\frac{1}{3}}}\right)}\right].$$

Hence $x=a-\frac{1}{8}\left[c^{\frac{1}{3}}\pm\sqrt{\left(\frac{8a-c}{3c^{\frac{1}{3}}}\right)}\right]^3$; or transposing $\sqrt[3]{(a+x)}$, and proceeding as before,

$$x=\tfrac{1}{8}\left[c^{\frac{1}{3}}\pm\sqrt{\left(\frac{8a-c}{3c^{\frac{1}{3}}}\right)}\right]^3-a.$$

Messrs. Bell and Cross let $y=\sqrt[3]{(a+x)}$, substitute, and then solve as above.

III. Solution by H. C. WHITAKER, M. Sc., Ph. D., Professor of Mathematics in the Manual Training School, Philadelphia, Pennsylvania.

For convenience in writing, denote $\sqrt[3]{c}$ by b, $\sqrt[3]{(a+x)}$ by y and $\sqrt[3]{(a-x)}$ by z. Of the following equations, (1) is given and the others are assumed. (Take $\alpha^3=1$).

$y+z-b=0$ . . . . (1).

$\alpha y+z-b=A$ . . . . (2).

$\alpha^2 y+z-b=B$ . . . . (3).

$y+\alpha z-b=C$ . . . . (4).

$\alpha y+\alpha z-b=D$ . . . . (5).

$\alpha^2 y+\alpha z-b=E$ . . . . (6).

$y+\alpha^2 z-b=F$ . . . . (7).

$\alpha y+\alpha^2 z-b=G$ . . . . (8).

$\alpha^2 y+\alpha^2 z-b=H$ . . . . (9).

We have, by multiplying these three at a time,

$$[y^3+(z-b)^3][y^3+(\alpha z-b)^3][y^3+(\alpha^2 z-b)^3]=0.$$

Or, completing the multiplications,

$$(y^3+z^3-b^3)^3+27b^3y^3z^3=0.$$

Restoring the original values of y, z, and b, we get,

$$27c(x^2-a^2)=(2a-c)^3.$$

Hence $x=\sqrt{\left(\dfrac{(2a-c)^3+27a^2c}{27c}\right)}$.

This may be the root of the given equation or the root of any of the assumed equations, depending on the various values of a and c.

[This example is found in Bonnycastle's Algebra (1845), page 97.]
Also solved by *P. S. BERG*, *H. C. WILKES*, and *G. B. M. ZERR*.

71. Proposed by **F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.**

When $x=0$, find the limit of the expression

$$u=\left(\frac{m+x}{m-x}\right)^{\frac{1}{x}}+\left(\frac{m-x}{m+x}\right)^{\frac{1}{x}}.$$

I. Solution by O. W. ANTHONY, M. Sc., Columbian University, Washington, D. C., and G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Let $u=u_1+u_2$. $\therefore$ $u_1=\left(\dfrac{m+x}{m-x}\right)^{\frac{1}{x}}$,

$$\log u_1=(1/x)\{\log(m+x)-\log(m-x)\}$$

$$=(1/x)\{[\log m+(x/m)-(x^2/2m^2)+(x^3/3m^3)-\ldots\ldots]-$$

$$[\log m-(x/m)-(x^2/2m^2)-(x^3/3m^3)-\ldots\ldots]\}$$

$$=\frac{2}{x}\left(\frac{x}{m}+\frac{x^3}{3m^3}+\frac{x^5}{5m^5}+\ldots\ldots\right)=2\left(\frac{1}{m}+\frac{x^2}{3m^2}+\frac{x^4}{5m^5}+\ldots\ldots\right)$$

$=2/m$, when $x=0$. $\log u_2=-\log u_1=-2/m$, when $x=0$.

$\therefore$ $u_1=e^{2/m}$, $u_2=e^{-2/m}$. $\therefore$ $u=e^{2/m}+e^{-2/m}$ when $x=0$.

II. Solution by H. C. WHITAKER, M. Sc., Ph. D., Professor of Mathematics in Philadelphia Manual Training School, Philadelphia, Pennsylvania.

Since $(1+x)^{1/x}=e$ when $x=0$, we have

$$\left(\frac{m+x}{m-x}\right)^{\frac{1}{x}}=\left[\left(1+\frac{2x}{m-x}\right)^{\frac{m-x}{2x}}\right]^{\frac{2}{m-x}}=e^{2/m}\text{ when }x=0.$$

In the same way $\left(\frac{m-x}{m+x}\right)^{\frac{1}{x}}=e^{-2m}$ when $x=0$. Hence $u=e^{2m}+e^{-2m}$.

Also solved by *J. SCHEFFER.*

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

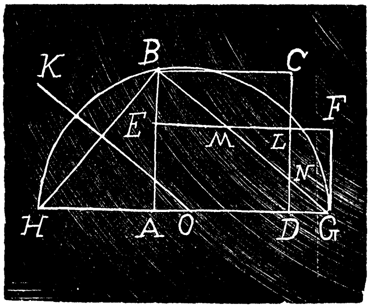
SOLUTIONS OF PROBLEMS.

69. Proposed by WILLIAM SYMMONDS, M. A., Professor of Mathematics and Astronomy in Pacific College, Santa Rosa, California ; P. O., Sebastopol, California.

To divide a square card into right-lined sections in a manner, that a rectangle of a given breadth can be formed from the sections; likewise, form a square from a rectangular card.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

(1). Let $ABCD$ be the square. Produce DA to H making AH equal the given width of the rectangle, join HB, and draw KO perpendicular to HB at its mid-point, then O is the center of the circle through HB. Produce AD to meet circle at G; AG is the length of the required rectangle. Take $AE=AH$ and complete the rectangle $AEFG$.



Now the right triangle AHB=right triangle BCN=right triangle MFG.

$\therefore CN=AE$ and $DN=BE$;

$\therefore \triangle BEM=\triangle DNG$.

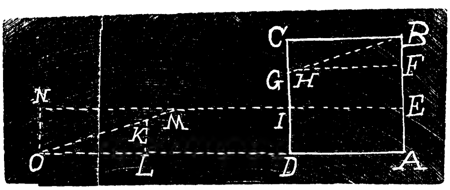
$\therefore ABCD=ADNME+BCN+BEM$ $=ADNME+MFG+NDG=AEFG$.

(2). Let $AEFG$ be the given rectangle. Produce GA to H making $AH=AE$. Upon HG describe the semi-circle. Then AB is a side of the required square. Complete the square $ABCD$ and draw BG. The rest of the proof is the same as above.

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

(1). Let $ABCD$ represent the square card. From A lay off on AB the the width of the rectangle successively as many times as possible, as AE, EF.

Then from the opposite corner C, lay off *one width only* of the rectangle, as CG. Now cut through on line GB. Then cut FH and EI parallel to DA.



Then will $EFGHI$ coincide with $DIMKL$, FBH with LKO, and BCG with MNO; and we have the rectangle $AENO$, of the given width AE, equivalent to the square $ABCD$.

(2). Let $AENO$ represent a rectangular card. Find the side of a square equivalent to the given rectangle. (Any geometry will show construction.)

Now from A lay off on OA the side of the square successively as many times as possible, as AD, DL. From N, lay off $NM=AD$.

Now, cut through on line OM. Then cut LK and DI perpendicular to AO.

Then will the sections of rectangle form a square as shown in first part of solution. The proof is evident.

III. Solution by C. H. WILSON, New York, New York.

Let $ABCD$ be the given square, and EF the breadth of the required rectangle. Find GH, a third proportional to EF and AB.

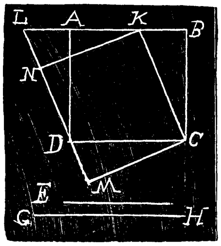


Fig. 1.

With C as a center and radius equal to GH, describe an arc cutting AB at K. (If $GH<BC$, use EF as radius). Cut off the triangle CBK and attach it in the position DAL. Draw KN perpendicular to LD. Cut of $\triangle LKN$ and attach it in the position DCM. $NKCM$ is the required rectangle, since it is equivalent to the square, and its area equals $NM\times NK=GH\times NK$. But by construction $GH\times EF=\overline{AB}^2$. $\therefore$ $EF=NK$ (Ax).

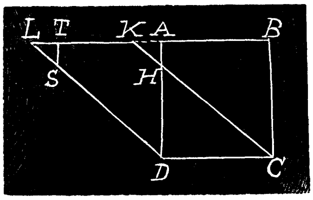


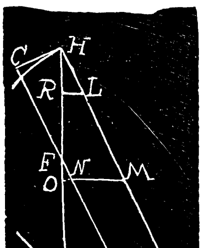
Fig. 2.

If K falls beyond A, attach $ARCB$ in position $TSDA$, cut off KAR and attach it in position LTS.

Then proceed as before.

Reverse operation (Fig. 1.) by finding mean proportional between MN and MC. With it as a radius and C as a center, describe an arc cutting MN in D. Cut off $\triangle MDC$ and attach it as NLK. Draw DA perpendicular to KL, cut off $\triangle LAD$ and attach it as KBC.

IV. Solution by the PROPOSER.



On AC, a side of the given square $ABCD$, draw a semi-circle AEC. Measure from A the chord AE equal to the given breadth of the rectangle. Produce CE, marking AB in S. On CD take CM, MN, etc., equal to AS.

Through m, n, etc., draw ml, nr, etc., parallel to CS. These will trace the lines of division required.

From the figure it is plain the rectangle $AEGH$ can be formed by joining fragments, AEC, AES, SM, m, Drn.

(2). On one of the longer sides AH of the given rectangle $AEGH$ describe the semi-circle ABH.

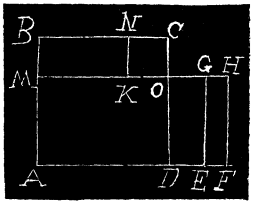
From A lay off chord AB equal in length to side of required square.

Join BH. Take BO, OR, etc., equal to AB. Through O, R, etc., draw OM, RL, etc., parallel to BA, marking the lines of division MN, LR, etc.

Hence the square $ABCD$ forms the parts of the rectangle, AES, SM, MR, RLH, HGF.

Corollary. When AS is greater than AB, or conversely, when AB is less than AS, the construction is quite simple.

V. Solution by J. W. SCROGGS, Principal of Rogers Academy, Rogers, Arkansas.



Let $ABCD$ be the given square, $AB=a+b$, $AM=MK=OD=GE=a$, $OK=OC=DE=b$, and $EF=b/r$.

Then area of $ADOM=a^2+ab$, and area of $BNKM=ab$.

$\therefore$ Their sum$=a^2+2ab$.

Let $b/a=r$. Then $a=br$, and area of $EFHG=b/r\times br=b^2$. $\therefore$ $AFHM=ABCD$.

CALCULUS.

Conducted by J. M, COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

49. Proposed by B. F. BURLESON, Oneida Castle, New York.

Find (1) in the leaf of the strophoid whose axis is a the axis of an inscribed leaf of the lemniscata, the node of the former coinciding with the crunode of the latter. Find (2) in a leaf of the lemniscata whose axis b the axis of a of an inscribed leaf of the strophoid, the node of the former also coinciding with the crunode of the latter.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Case I.

The equation of strophoid with origin at node is,

$$y^2=\frac{x(x-a)^2}{2a-x}\dots\dots\dots\dots\dots\dots\dots\dots(1).$$

The equation of lemniscate with origin at crunode is,

$$(x^2+y^2)^2=b^2(x^2-y^2)\dots\dots\dots\dots\dots\dots\dots\dots(2).$$

In order that the latter may be inscribed in the former we must have tangency. $\therefore$ x, y, and dy/dx must be equal for each curve.

Substituting y from (1) in (2) we get after reduction,

$$x^3-4ax^2+\left(\frac{9a^2b^2-a^4}{2b^2}\right)x-a^3=0 \quad \ldots\ldots\ldots\ldots\ldots (3).$$

$$\frac{dy}{dx}=\frac{b^2x-2x^3-2xy^2}{y(b^2+2x^2+2y^2)}=\frac{4ax^2-4a^2x+a^3-x^3}{y(2a-x)^2} \quad \ldots\ldots\ldots\ldots (4).$$

Substituting y from (1) in (4) and reducing we get,

$$x^4-6ax^3+12a^2x^2-\left(\frac{17a^3b^2-2a^5}{2b^2}\right)x+a^4=0 \quad \ldots\ldots\ldots\ldots (5).$$

(5) may be written as follows :

$$x^4-5ax^3+8a^2x^2-\left(\frac{8a^3b^2-a^5}{2b^2}\right)x-a\left\{x^3-4ax^2+\left(\frac{9a^2b^2-a^4}{2b^2}\right)x-a^3\right\}=0.$$

(3) in the last equation gives,

$$x^3-5ax^2+8a^2x-\frac{8a^3b^2-a^5}{2b^2}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

(3)−(6) gives, $x^2-\dfrac{7ab^2+a^3}{2b^2}x-\dfrac{a^4-6a^2b^2}{2b^2}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (7).$

$\dfrac{8b^2-a^2}{2b^2}$ times (3)−(6) gives,

$$(6b^2-a^2)x^2-(22ab^2-4a^3)x+\frac{40a^2b^4-17a^4b^2+a^6}{2b^2}=0 \quad \ldots\ldots\ldots (8).$$

$(6b^2-a^2)$ times (7)−(8) gives,

$$x=\frac{4ab^4-5a^3b^2}{2b^4-7a^2b^2+a^4} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

$\dfrac{40a^2b^4-17a^4b^2+a^6}{2b^2}$ times (7)$+\dfrac{a^4-6a^2b^2}{2b^2}$ times (8) gives,

$$x=\frac{16ab^6+13a^3b^4-18a^5b^2+a^7}{8b^6-10a^2b^4} \quad \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

From (9) and (10) we get,

$$6b^8+161a^2b^6-141a^4b^4+25a^6b^2-a^8=0 \quad \ldots\ldots\ldots\ldots\ldots (11).$$

Let $a^2/b^2=u$, then (11) becomes,

$$u^4-25u^3+141u^2-161u-6=0 \ldots\ldots\ldots\ldots (12).$$

$\therefore$ $u=1.586892$. $\therefore$ $a^2=1.586892b^2$.

$\therefore$ $a=1.2597b$. $\therefore$ $b=.7938a$.

CASE II.

The equation of the strophoid with origin at crunode is,

$$y^2=\frac{x^2(a+x)}{a-x} \ldots\ldots\ldots\ldots (13).$$

The equation of the lemniscate with origin at node is,

$$\{(x+b)^2+y^2\}^2=b^2\{(x+b)^2-y^2\} \ldots\ldots\ldots\ldots (14).$$

In order that the former may be inscribed in the latter we must have x, y, dy/dx equal for both curves.

(13) in (14) gives,

$$(2a^2+b^2-4ab)x^3+(4a^2b-5ab^2+b^3)x^2+(4a^2b^2-2ab^3)x+a^2b^3=0 \ldots\ldots (15).$$

$$\frac{dy}{dx}=\frac{a^2x+ax^2-x^3}{y(a-x)^2}=-\frac{2x^3+6bx^2+5b^2x+b^3+2(x+b)y^2}{y(2x^2+4bx+3b^2+2y^2)} \ldots\ldots (16).$$

(13) in (16) gives,

$$(8ab-4a^2-2b^2)x^4+(8a^3-20a^2b+9ab^2-b^3)x^3$$
$$+(12a^3b-15a^2b^2+3ab^3)x^2+(8a^3b^2-3a^2b^3)x+a^3b^3=0 \ldots (17).$$

(17) may be written

$$(8ab-4a^2-2b^2)x^4+(6a^3-16a^2b+8ab^2-b^3)x^3$$
$$+(8a^3b-10a^2b^2+2ab^3)x^2+(4a^3b^2-a^2b^3)x$$
$$+a[(2a^2+b^2-4ab)x^3+(4a^2b-5ab^2+b^3)x^2+(4a^2b^2-2ab^3)x+a^2b^3]=0 \ldots (18).$$

(15) in (18) gives,

$$2(2a^2+b^2-4ab)x^3+(16a^2b-8ab^2+b^3-6a^3)x^2$$
$$+(10a^2b^2-8a^3b-2ab^3)x+a^2b^3-4a^3b^2=0 \ldots (19).$$

(19)—2 times (15) gives,

$$(8a^2b+2ab^2-b^3-6a^3)x^2+(2a^2b^2-8a^3b+2ab^3)x-a^2b^3-4a^3b^2=0 \quad ......(20).$$

b times (19)—($b-4a$) times (15) gives,

$$(8a^3+b^3-14a^2b)x^2+(ab^3+10a^3b-8a^2b^2)x+8a^3b^2-2a^2b^3=0 \quad ....(21).$$

$(8a^3+b^3-14a^2b)$ times (20)—$(8a^2b+2ab^2-b^3-6a^3)$ times (21) gives,

$$x=\frac{14a^3b^3+16a^5b+8a^2b^4-28a^4b^2-3ab^5}{4a^5-38a^3b^2+8ab^4+18a^2b^3-3b^5} \quad ....(22).$$

$(8a-2b)$ times (20)+$(4a+b)$ times (21) gives,

$$x=\frac{10a^3b^2-24a^4b+8a^2b^3-3ab^4}{14a^2b^2+16a^4+8ab^3-28a^3b-3b^4} \quad ....(23).$$

From (22) and (23) we get,

$$9b^5+16ab^4-148a^2b^3-144a^3b^2+468a^4b-176a^5=0 \quad ....(24).$$

Let $b/a=v$, then (24) becomes,

$$9v^5+16v^4-148v^3-144v^2+468v-176=0.$$

$\therefore\ v=1.12257.\quad \therefore\ b=1.12257a.\quad \therefore\ a=.8908b.$

The published solution assumes that both curves coincide at their crunodes. This is not what the problem calls for. The above solution realizes in every respect the demands of the problem.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

44. Proposed by O. W. ANTHONY, M. Sc., Columbian University, Washington, D. C.

There is a triangle whose sides repulse a center of force within the triangle with an intensity that varies inversely as the distance of the center of force from each point of the sides of the triangle. What is the position of equilibrium of the center?

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

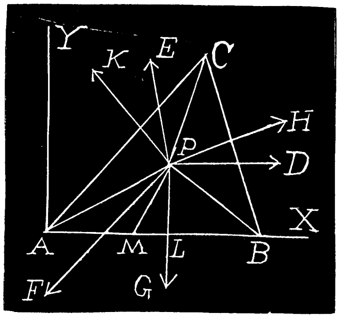
Let $BC=a$, $AC=b$, $AB=c$, $AP=h$, $BP=k$, $CP=l$, $AL=m$, $PL=n$, $\angle APB=\beta$, $\angle BPC=\gamma$, $\angle APC=\delta$.

Then $\beta=\cos^{-1}\left(\dfrac{k^2+h^2-c^2}{2hk}\right)$, $\gamma=\cos^{-1}\left(\dfrac{k^2+l^2-a^2}{2kl}\right)$, $\delta=\cos^{-1}\left(\dfrac{h^2+l^2-b^2}{2hl}\right)$,

$$h^2=m^2+n^2,\ k^2=(c-m)^2+n^2,\ l^2=(b\cos A-m)^2+(b\sin A-n)^2.$$

If P be the required point, A origin, AX, AY, axes, $AM=x$, ρ=density and $\varkappa$ area of cross section of the lines perpendicular to their length, ν=mass of P, μ=some constant, then the attraction on M is,

$$\frac{\mu\nu\rho\varkappa dx}{PM}=\frac{\mu\nu\rho\varkappa dx}{\{n^2+(m-x)^2\}^{\frac{1}{2}}}.$$



Call $\mu\nu\rho\varkappa$, Q. Then the resolved part of the force parallel to axis of X is,

$$\frac{Q(m-x)dx}{n^2+(m-x)^2},$$

and the resolved part parallel to the axis of Y is,

$$\frac{Qndx}{n^2+(m-x)^2}.$$

$$\therefore\ PD=-Q\int_o^c\frac{(m-x)dx}{n^2+(m-x)^2}=\tfrac{1}{2}Q\log\left\{\frac{n^2+(c-m)^2}{n^2+m^2}\right\}.$$

$\therefore\ PD=Q\log(k/h)$, $PE=Q\log(l/k)$, $PF=Q\log(h/l)$.

$$PG=-Q\int_o^c\frac{ndx}{n^2+(m-x)^2}=-Q\{\tan^{-1}(m/n)+\tan^{-1}\frac{c-m}{n}\}.$$

$\therefore\ PG=-Q\beta$, $PH=-Q\gamma$, $PK=-Q\delta$.

If the resultant of two forces is equal to the third, the three forces are in equilibrium.

Regarding PD, PE, PF as three forces and PG, PH, PK as three forces, we get,

$$\left.\begin{array}{l}\{\log(k/h)\}^2+\{\log(l/k)\}^2+2\log(k/h)\log(l/k)\cos B=\{\log(h/l)\}^2\\ \beta^2+\gamma^2-2\beta\gamma\sin B=\delta^2\end{array}\right\}\ \ldots\ldots\ldots\ldots(1).$$

$$\left.\begin{array}{l}\{\log(l/k)\}^2+\{\log(h/l)\}^2+2\log(l/k)\log(h/l)\cos C=\{\log(k/h)\}^2\\ \quad\gamma^2+\delta^2-2\gamma\delta\sin C=\beta^2\end{array}\right\}\ldots\ldots\ldots\ldots(2).$$

$$\left.\begin{array}{l}\{\log(h/l)\}^2+\{\log(k/h)\}^2+2\log(h/l)\log(k/h)\cos A=\{\log(l/k)\}^2\\ \quad\beta^2+\delta^2-2\beta\delta\sin A=\gamma^2\end{array}\right\}\ldots\ldots\ldots\ldots(3).$$

Substituting the values of h, k, l, β, γ, δ in terms of m, n in either (1), (2), or (3), we get, in either case, two equations in m and n from which, if possible, the values of m and n may be found and the point P determined.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

80. Proposed by CHARLES C. CROSS, Laytonsville, Maryland.

From a cask containing 10 gallons of wine, a servant drew off 1 gallon each day, for five days, each time supplying the deficiency by adding a gallon of water. Afterwards, fearing detection, he again drew off a gallon a day for five days, adding each time a gallon of wine. How many gallons of water still remained in the cask? [From *Quackenbos' Arithmetic*.]

81. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

How far will a body fall in the first second on the sun, the density of the sun being 25 times that of the earth and its diameter 866400 miles?

82. Proposed by CHARLES C. CROSS, Laytonsville, Maryland.

Two men, A and B, started from the same point at the same time; A traveled southeast for 10 hours and at the rate of 10 miles per hour, and B due south for the same time, going 6 miles per hour; they then turned and traveled directly towards each other at the same rates respectively, till they met. How far did each man travel?

DIOPHANTINE ANALYSIS.

55. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

Construct a general Magic Square whose sum is $3m$.

56. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

If $\phi(R)$ is the number of integers which are less than R and prime to it, and if y is prime to R, show that $y^{\phi(R)}-1\equiv0(\mathrm{mod}.R)$.

57. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

Each of *five* of the digits may be the terminal figure of a perfect integral square. Each of *eighteen* combinations of two digits may be the *two* terminal figures of an integral square. Each of *one hundred and nineteen* combinations of three digits may be the *three* terminal figures of an integral square. *Under these conditions*, what is the greatest number of arrangements of the nine digits, all taken together, whose three terminal figures shall be those of a square number?

MISCELLANEOUS.

53. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

(a). What is the highest north latitude in which the sun will shine in at the north window of a building at least once in a year?

(b). How many days will it shine in at the north window of a building in latitude 41° N.?

54. Proposed by S. HART WRIGHT, M. D., A. M., Ph. D., Penn Yan, New York.

On latitude 40° N.$=\gamma$, when the moon's declination is 5° 23′ N.$=\delta$, and the sun's 9° 52′ S.$=-\delta$, how long after sunset, will the two horns or cusps of the moon's crescent (recently new) set at the same moment, the crescent with its back *down* having touched the horizon first? Semi-diameters, refraction, and parallax not considered.

55. Proposed by J. M. COLAW, A. M., Monterey, Virginia.

Multiply 6 by 4. Is the problem legitimate when both symbols represent pure number?

[NOTE. "A measured or numbered quantity may be divided into a number of parts, or taken a number of times; but no number can be multiplied or divided into parts."—*McLellan and Dewey's Psychology of Number*. "The astounding thesis is maintained that number is not a magnitude, does not possess quantity at all, and that 'no number can be multiplied or divided into parts'."—*Lefevre's Number and Its Algebra*.]

BOOKS.

Theory of Discrete Manifoldness. By F. W. Franklin. Pamphlet, 12 pages. Published by the Author.

Recent Books on Quaternions (from "Science," Vol. V, pages 699—701). By Dr. Alexander Macfarlane.

This is a concise criticism on the following works: *Theorie der Quaternionen*, Von Dr. P. Molenbroke; *Anwendung der Quaternionen auf die Geometrie*, by same author; *The Oatlines of Quaternions*, by Lieut. Col. H. W. L. Hime; *A Primer of Quaternions*, by A. S. Hathaway; and *Utility of Quaternions in Physics*, by A. McAulay. B. F. F.

Application of Hyberbolic Analysis to the Discharge of a Condenser. By Dr. Alexander Macfarlane. Pamphlet, 16 pages.

A paper presented at the annual meeting of the American Institute of Electrical Engineers, May 18th, 1897. B. F. F.

Introduction to American Literature, including Illustrative Selections with Notes. By F. V. N. Painter, A. M., D. D., Professor of Modern Languages in Roanoke College, Author of a History of Education, Introduction to English Literature, etc. 8vo. Cloth, 498 pages. Boston: Leach, Shewell and Sanborn.

This is the best prepared work on American Literature with which we are acquainted. Only the very best productions from the best authors are selected. The selections for special notice, which are chosen to illustrate the distinguishing characteristics of each author are supplied with explanatory notes. The short sketch of our leading writers is written in an intensely interesting manner, all the kernels having been preserved and all the chaff thrown away. Each sketch is preceded by an excellent portrait. B. F. F.

THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. IV. JUNE-JULY, 1897. No. 6-7.

BIOGRAPHY.

JAMES JOSEPH SYLVESTER, LL. D., F. R. S.

BY GEORGE BRUCE HALSTED.

ON Monday, March 15, 1897, in London, where, September 3, 1814, he was born, died the most extraordinary personage for half a century in the mathematical world.

James Joseph Sylvester was second wrangler at Cambridge in 1837. When we recall that Sylvester, Wm. Thomson, Maxwell, Clifford, J. J. Thomson were all second wranglers, we involuntarily wonder if any senior wrangler except Cayley can be ranked with them.

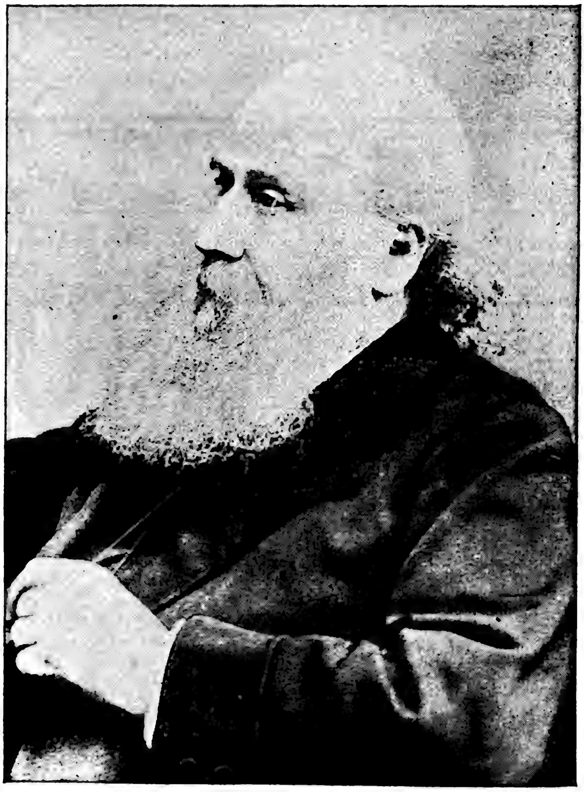
Yet it was characteristic of Sylvester that not to have been first was always bitter to him.

The man who beat him, Wm. N. Griffin, also a Johnion, afterwards a modest clergyman, was tremendously impressed by Sylvester, and honored him in a treatise on optics where he used Sylvester's first published paper, "Analytical development of Fresnel's optical theory of crystals," *Philosophical Magazine*, 1837.

Sylvester could not be equally generous, and explicitly rated above Griffin the fourth wrangler George Green, justly celebrated, who died in 1841.

Sylvester's second paper, "On the motion and rest of fluids," *Philosophical Magazine*, 1838 and 1839, also seemed to point to physics.

In 1838 he succeeded the Rev. Wm. Ritchie as professor of natural philosophy in University College, London.



JAMES JOSEPH SYLVESTER.

His unwillingness to submit to the religious tests then enforced at Cambridge and to sign the 39 articles not only debarred him from his degree and from competing for the Smith's prizes, but, what was far worse, deprived him of the Fellowship morally his due. He keenly felt the injustice.

In his celebrated address at the Johns Hopkins University his denunciation of the narrowness, bigotry and intense selfishness exhibited in these compulsory creed tests, made a wonderful burst of oratory. These opinions were fully shared by De Morgan, his colleague at University College. Copies I possess of the five examination papers set by Sylvester at the June examination, session of 1839-40, show him striving as a physicist, but it was all a false start. Even his first paper shows he was always the Sylvester we knew. To the "Index of Contents" he appends the characteristic note : "Since writing this index I have made many additions more interesting than any of the propositions here cited, which will appear toward the conclusion." Ever he is borne along helpless but ecstatic in the ungovernable flood of his thought.

A physical experiment never suggests itself to the great mental experimenter. Cayley once asked for his box of drawiug instruments. Sylvester answered, "I never had one." Something of this irksomeness of the outside world, the world of matter, may have made him accept, in 1841, the professorship offered him in the University of Virginia.

On his way to America he visited Rowan Hamilton at Dublin in that observatory where the maker of quaternions was as out of place as Sylvester himself would have been. The Virginians so utterly failed to understand Sylvester, his character, his aspirations, his powers, that the Rev. Dr. Dabney, of Virginia, has seriously assured me that Sylvester was actually deficient in intellect, a sort of semi-idiotic calculating boy. For the sake of the contrast, and to show the sort of civilization in which this genius had risked himself, two letters from Sylvester's tutors at Cambridge may here be of interest.

The great Colenso, Bishop of Natal, previously Fellow and Tutor of St. John's College, writes : "Having been informed that my friend and former pupil, Mr. J. J. Sylvester, is a candidate for the office of professor of mathematics, I beg to state my high opinion of his character both as a mathematician and a gentleman.

"On the former point, indeed, his degree of Second Wrangler at the University of Cambridge would be, in itself, a sufficient testimonial. But I beg to add that his powers are of a far higher order than even that degree would certify."

Philip Kelland, himself a Senior Wrangler, and then professor of mathematics in the University of Edinburgh, writes : "I have been requested to express my opinion of the qualifications of Mr. J. J. Sylvester, as a mathematician.

"Mr. Sylvester was one of my private pupils in the University of Cambridge, where he took the degree of Second Wrangler. My opinion of Mr. Sylvester then was that in originality of thought and acuteness of perception he had never been surpassed, and I predicted for him an eminent position among the mathematicians of Europe. My anticipations have been verified. Mr. Sylves-

ter's published papers manifest a depth and originality which entitles them to the high position they occupy in the field of scientific discovery. They prove him to be a man able to grapple with the most difficult mathematical questions and are satisfactory evidence of the extent of his attainments and the vigor of his mental powers."

The five papers produced in this year, 1841, before Sylvester's departure for Virginia, show that now his key note is really struck. They adumbrate some of his greatest discoveries.

They are: "On the relation of Sturm's auxiliary functions to the roots of an algebraic equation," British Assoc. Rep. (pt. 2), 1841; "Examples of the dialytic method of elimination as applied to ternary systems of equations," Camb. M. Jour. II., 1841; On the amount and distribution of multiplicity in an algebraic equation," Phil. Mag. XVII., 1841; "On a new and more general theory of multiple roots," Phil. Mag. XVIII., 1841; "On a linear method of eliminating between double, treble and other systems of algebraic equations," Phil. Mag. XVIII., 1841; "On the dialytic method of elimination," Phil. Mag. XXI., Irish Acad. Proc. II.

This was left behind in Ireland, on the way to Virginia. Then suddenly occurs a complete stoppage in this wonderful productivity. Not one paper, not one word, is dated from the University of Virginia. Not until 1844 does the wounded bird begin again feebly to chirp, and indeed it is a whole decade before the song pours forth again with mellow vigor that wins a waiting world.

Disheartening was the whole experience; but the final cause of his sudden abandonment of the University of Virginia I gave in an address entitled, "Original Research and Creative Authorship the Essence of University Teaching," printed in *Science*, N. S., Vol. I., pp. 203-7, February 22, 1895.

On the return to England with heavy heart and dampened ardor, he takes up for his support the work of an actuary and then begins the study of law. In 1847 we find him at 26 Lincoln's Inn Fields, "eating his terms." On November 22, 1850, he is called to the bar and practices conveyancing.

But already in his paper dated August 12, 1850, we meet the significant names Boole, Cayley, and harvest is at hand.

The very words which must now be used to say what had already happened and what was now to happen were not then in existence. They were afterward made by Sylvester and constitute in themselves a tremendous contribution. As he himself says: "Names are, of course, all important to the progress of thought, and the invention of a really good name, of which the want, not previously perceived, is recognized, when supplied, as having ought to be felt, is entitled to rank on a level in importance, with the discovery of a new scientific theory."

Elsewhere he says of himself: "Perhaps I may without immodesty lay claim to the appellation of the Mathematical Adam. as I believe that I have given more names (passed into general circulation) to the creatures of the mathematical reason than all the other mathematicians of the age combined."

In one year, 1851, Sylvester created a whole new continent, a new world in the universe of mathematics. Demonstration of its creation is given by the Glossary of New Terms which he gives in the *Philosophical Transactions*, Vol. 143, pp. 543-548.

Says Dr. W. Franz Meyer in his exceedingly valuable Bericht über die Fortschritte der projectiven Invariantentheorie, the best history of the subject (1892):

"Als äusseres Zeichen für den Umfang der vorgeschrittenen Entwickelung mag die ausgedehnte, grösstenteils von *Sylvester* selbst herrührende Terminologie dienen, die sich am Ende seiner grossen Abhandlung über Sturm'sche Functionen (1853) zusammengestellt findet."

Using then this new language, let us briefly say what had happened in the decade when Sylvester's genius was suffering from its Virginia wound. The birth-day of the giant *Theory of Invariants* is April 28, 1841, the date attached by George Boole to a paper in the Cambridge *Mathematical Journal* where he not only proved the invariantive property of discriminants generally, but also gave a simple principle to form simultaneous invariants of a system of two functions. The paper appeared in November, 1841, and shortly after, in February, 1842, Boole showed that the polars of a form lead to a broad class of covariants. Here he extended the results of the first article to more than two Forms. Boole's papers led Cayley, nearly three years later (1845), to propose to himself the problem to determine *a priori* what functions of the coefficients of an equation possess this property of invariance, and he discovered its possession by other functions besides discriminants, for example the quadrinvariants of binary quantics, and in particular the invariant S of a quartic.

Boole next discovered the other invariant T of a quartic and the expression of the discriminant in terms of S and T. Cayley next (1846) published a symbolic method of finding invariants. Early in 1851 Boole reproduced, with additions, his paper on Linear Transformations; then at last began Sylvester. He always mourned what he called "the years he lost fighting the world"; but, after all, it was he who made the Theory of Invariants.

Says Meyer: "sehen wir in dem Cyklus *Sylvester*'scher Publicationen (1851–1854) bereits die Grundzüge einer allgemeinen Theorie erstehen, welche die Elemente von den verschiedenartigsten Zweigen der späteren Disciplin umfasst." "Sylvester beginnt damit, die Ergebnisse seiner Vorgänger unter einem einzigen Gesichtspunkte zuvereinigen."

With deepest foresight Sylvester introduced, together with the original variables, those dual to them, and created the theory of contravariants and intermediate forms. He introduced, with many other processes for producing invariantive forms, the principle of mutual differentiation.

Hilbert attributes the sudden growth of the theory to these processes for producing and handling invariantive creatures. "Die Theorie dieser Gebilde erhob sich, von speciellen Aufgaben ausgehend, rasch zu grosser Allgemeinheit —dank vor Allem dem Umstande, dass es gelang, eine Reihe von besonderen der Invariantentheorie eigenthümlichen Prozessen zu entdecken, deren An-

wendung die Aufstellung und Behandlung invarianter Bildungen beträchtlich erleichterte."

"Was die Theorie der algebraischen Invarianten anbetrifft so sind die ersten Begründer derselben, *Cayley* und *Sylvester*, zugleich auch als die Vertreter der naiven Periode anzusehen : an der Aufstellung der einfachsten Invariantenbildungen und an den eleganten Anwendungen auf die Auflösung der Gleichungen der ersten 4 Grade hatten sie die unmittelbare Freude der ersten Entdeckung." It was Sylvester alone who created the theory of canonic forms and proceeded to apply it with astonishing power. What marvelous mass of brand new being he now brought forth!

Moreover he trumpeted abroad the eruption. He called for communications to himself in English, French, Italian, Latin or German, so only the "Latin character" were used.

From 1851 to 1854 he produces forty-six different memoirs. Then comes a dead silence of a whole year, broken in 1856 by a feeble chirp called "A Trifle on Projectiles."

What has happened? Some more "fighting the world." Sylvester declared himself a candidate for the vacant professorship of geometry in Gresham College, delivered a probationary lecture on the 4th of December, 1854, and was ignominiously "turned down." Let us save a couple of sentences from this lecture:

"He who would know what geometry is must venture boldly into its depths and learn to think and feel as a geometer. I believe that it is impossible to do this, to study geometry as it admits of being studied, and I am conscious it can be taught, without finding the reasoning invigorated, the invention quickened, the sentiment of the orderly and beautiful awakeded and enhanced, and reverence for truth, the foundation of all integrity of character, converted into a fixed principle of the mental and moral constitution, according to the old and expressive adage '*abeunt studia in mores.*' "

But this silent year concealed still another stunning blow of precisely the same sort, as bears witness the following letter from Lord Brougham to The Lord Panmure:

"BROUGHAM,

PRIVATE. 28 Aug. 1855.

MY DEAR P.

My learned excellent friend and brother mathematician Mr. Sylvester is again a candidate for the professorship at Woolwich on the death of Mr. O'Brian who carried it against him last year.

I entreat once more your favorable consideration of this eminent man who has already to thank you for your great kindness.

Yours sincerely,

H. BROUGHAM.

On this third trial, backed by such an array of credentials as no man ever presented before, he barely scraped through, was appointed professor of mathematics at the Royal Military Academy, and served at Woolwich exactly 14 years, 10 months, and 15 days.

A single sentence of his will best express his greatest achievement there and his manner of exit thence:

"If Her most Gracious Majesty should ever be moved to recognize the palmary exploit of the writer of this note in the field of English science as having been the one successfully to resolve a question and conquer an algebraical difficulty which had exercised in vain for two centuries past, since the time of Newton, the highest mathematical intellects in Europe (Euler, Lagrange, Maclaurin, Waring among the number), by conferring upon him some honorary distinction in commemoration of the deed, he will crave the privilege of being allowed to enter the royal presence, not covered, like De Courcy, but barefooted, with rope around his waist, and a *goose*-quill behind his ear, in token of repentant humility, and as an emblem of convicted simplicity in having once supposed that on such kind of success he could found any additional title to receive fair and just consideration at the hands of Her Majesty's Government when quitting his appointment as public professor at Woolwich under the coercive operation of a non-Parliamentary retrospective and utterly unprecedented War Office enactment." Athenæum Club, January 31, 1871. Of course this means a row of barren years, 1870, 1871, 1872, 1873.

The fortunate accident of a visit paid Sylvester in the autumn of 1873 by Pafnuti Lvovich Chebyshev, of the Universty of St. Petersburg, reawakened our genius to produce in a single burst of enthusiasm a new branch of science.

On Friday evening, January 23, 1874, Sylvester delivered at the Royal Institution a lecture entitled "On Recent Discoveries in Mechanical Conversion of Motion," whose ideas, carried on by two of his hearers, H. Hart and A. B. Kempe, have made themselves a permanent place even in the elements of geometry and kinematics. A synopsis of this lecture was published, but so curtailed and twisted into tha third person that the life and flavor are quite gone from it. I possess the unique manuscript of this epoch-making lecture as actually delivered. A few sentences will show how characteristic and inimitable was the original form:

"The air of Russia seems no less favorable to mathematical acumen than to a genius for fable and song. Lobacheffsky, the first to mitigate the severity of the Euclidean code and to beat down the bars of a supposed adamantine necessity, was born (a Russian of Russians), in the government of Nijni Novgorod; Tchebicheff [Chebyshev], the prince and conqueror of prime numbers, able to cope with their refractory character and to confine the stream of their erratic flow, their progression, within algebraic limits, in the adjacent circumscription of Moscow; and our own Cayley was cradled amidst the snows of St. Petersburg." [Sylvester himself contracted Chebyshev's limits for the distribution of primes.] "I think I may fairly affirm that a simple direct solution of the problem of the duplication of the cube by mechanical means was never accomplished down to this day. I will not say but that, by a merciful interpretation of his oracle, Apollo may have put up with the solution which the ancient geometers obtained by means of drawing two parabolic curves; but of this I feel assured that had I been

then alive, and could have shown my solution, which I am about to exhibit to you, Apollo would have leaped for joy and danced (like David before the ark), with my triple cell in hand, in place of his lyre, before his own duplicated altar."

That in the very next year Sylvester was taking a more active part than has hitherto been known in the organization of the incipient Johns Hopkins University is seen from the following letter to him in London from the great Joseph Henry:

SMITHSONIAN INSTITUTION,
August 25, 1875.

MY DEAR SIR:

Your letter of the 13th inst. has just been received and in reply I have to say that I have written to President Gilman of the Hopkins University giving my views as to what it ought to be and have stated that if properly managed it may do more for the advance of literature and science in this country than any other institution ever established; it is entirely independent of public favor and may lead instead of following popular opinion.

I have advised that liberal salaries be paid to the occupants of the principal chairs and that to fill them the best men in the world who can be obtained should be secured.

I have mentioned your name prominently as one of the very first mathematicians of the day; what the result will be, however, I can not say.

The Trustees are all citizens of Baltimore and among them I have some personal friends; the President, Mr. Gilman, and one of them, came to Washington a few weeks ago to get from me any suggestions that I might have to offer.

It is to be regretted that in this country the Trustees, who control the management of bequests of this character, think it important to produce a palpable manifestation of the institution to be established by spending a large amount of the bequest in architectural displays. Against this custom I have protested and have asserted that if the proper men and the necessary implements of instruction are provided, the teaching may be done in log cabins.

It would give me great pleasure to have you again as my guest, and I will do what I can to secure your election.

Very truly your friend,
JOSEPH HENRY.

We know the result.

Sylvester was offered the place; demanded a higher salary; won; came.

I was his first pupil, his first class, and he always insisted that it was I who brought him back to the Theory of Invariative Forms. In a letter to me of September 24, 1882, he writes: "Nor can I ever be oblivious of the advantage which I derived from your well-grounded persistence in inducing me to lecture on the Modern Algebra, which had the effect of bringing my mind back to this subject, from which it had for some time previously been withdrawn, and in which I have been laboring, with a success which has considerably exceeded my anticipations, ever since."

He made this same statement at greater length in his celebrated address at the Johns Hopkins on February 22, 1877: "At this moment I happen to be engaged in a research of fascinating interest to myself, and which, if the day only responds to the promise of its dawn, will meet, I believe, a sympathetic response from the professors of our divine algebraical art wherever scattered through the world.

"There are things called Algebraical Forms; Professor Cayley calls them

Quantics. These are not, properly speaking, Geometrical Forms, although capable, to some extent, of being embodied in them, but rather schemes of processes, or of operations for forming, for calling into existence, as it were, algebraic quantities.

"To every such Quantic is associated an infinite variety of other forms that may be regarded as engendered from and floating, like an atmosphere, around it; but infinite in number as are these derived existences, these emanations from the parent form, it is found that they admit of being obtained by composition, by mixture, so to say, of a certain limited number of fundamental forms, standard rays, as they might be termed, in the Algebraic Spectrum of the Quantic to which they belong; and, as it is a leading pursuit of the physicists of the present day to ascertain the fixed lines in the spectrum of every chemical substance, so it is the aim and object of a great school of mathematicians to make out the fundamental derived forms, the Covariants and Invariants, as they are called, of these Quantics.

"This is the kind of investigation in which I have, for the last month or two, been immersed, and which I entertain great hopes of bringing to a successful issue.

"Why do I mention it here? It is to illustrate my opinion as to the invaluable aid of teaching to the teacher, in throwing him back upon his own thoughts and leading him to evolve new results from ideas that would have otherwise remained passive or dormant in his mind.

"But for the persistence of a student of this university in urging upon me his desire to study with me the modern algebra I should never have been led into this investigation; and the new facts and principles which I have discovered in regard to it (important facts, I believe) would, so far as I am concerned, have remained still hidden in the womb of time. In vain I represented to this inquisitive student that he would do better to take up some other subject lying less off the beaten track of study, such as the higher parts of the Calculus or Elliptic Functions, or the theory of Substitutions, or I wot not what besides. He stuck with perfect respectfulness, but with invincible pertinacity, to his point. He would have the New Algebra (Heaven knows where he had heard about it, for it is almost unknown on this continent), that or nothing. I was obliged to yield, and what was the consequence? In trying to throw light upon an obscure explanation in our text-book my brain took fire; I plunged with requickened zeal into a subject which I had for years abandoned, and found food for thoughts which have engaged my attention for a considerable time past, and will probably occupy all my powers of contemplation advantageously for several months to come."

Another specific instance of the same thing he mentions in his paper, "Proof of the Hitherto Undemonstrated Fundamental Theorem of Invariants," dated November 13, 1877:

"I am about to demonstrate a theorem which has been waiting proof for the last quarter of a century and upwards. It is the more necessary that this

should be done, because the theorem has been supposed to lead to false conclusions, and its correctness has consequently been impugned. Thus in Professor Faà de Bruno's valuable *Théorie des formes binaires*, Turin, 1876, at the foot of page 150 occurs the following passage: "Cela suppose essentiellement que les équations de condition *soient toutes indépendantes entr'elles, ce qui n'est pas toujours le cas*, ainsi qu'il résulte des recherches du Professor Gordan sur les nombres des covariants des formes quintique et sextique."

The reader is cautioned against supposing that the consequence alleged above does result from Gordan's researches, which are indubitably correct. This supposed consequence must have arisen from a misapprehension, on the part of M. de Bruno, of the nature of Professor Cayley's rectification of the error of reasoning contained in his second memoir on Quantics, which had led to results discordant with Gordan's. Thus error breeds error, unless and until the pernicious brood is stamped out for good and all under the iron heel of rigid demonstration. In the early part of this year Mr. Halsted, a fellow of Johns Hopkins University, called my attention to this passage in M. de Bruno's book; and all I could say in reply was that 'the extrinsic evidence in support of the independence of the equations which had been impugned rendered it in my mind as certain as any fact in nature could be, but that to reduce it to an exact demonstration transcended, I thought, the powers of the human understanding.' "

In 1883 Sylvester was made Savilian professor of geometry at Oxford, the first Cambridge man so honored since the appointment of Wallis in 1649.

To greet the new environment, he created a new subject for his researches —Reciprocants, which has inspired, among others, J. Hammond, of Oxford; McMahon, of Woolwich; A. R. Forsyth, of Cambridge; Leudesdorf, Elliott and Halphen.

Sylvester never solved exercise problems such as are proposed in the *Educational Times*, though he made them all his life long down to his latest years. For example, *unsolved* problems by him will be found even in Vol. LXII. and Vol. LXIII. of the *Educational Times* reprints (1895). If at the time of meeting his own problem he met also a neat solution he would communicate them together, but he never solved any. In the meagre notices that have been given of Sylvester the strangest errors abound. Thus C. S. Pierce, in the *Post*, March 16th, speaks of his accepting, "with much diffidence," a word whose meaning he never knew; and gives 1862 as the date of his retirement from Woolwich, which is eight years wrong, as this forced retirement was July 31, 1870, after his 55th birthday. Cajori, in his inadequate account (History of Mathematics, p. 326), puts the studying of law before the professorship at University College and the professorship at the University of Virginia, both of which it followed. Effect must follow cause. And strange, that of the few things he ascribes to Sylvester, he should have hit upon something not his, "the discovery of the partial differential equations satisfied by the invariants and covariants of binary quantics." But Sylvester has explicitly said in Section VI. of his "Calculus of Forms:" "I alluded to the partial differential equations by which every invariant may be de-

fined. M. Aronhold, as I collect from private information, was the first to think of the application of this method to the subject; but it was Mr. Cayley who communicated to me the equations which define the invariants of functions of two variables."

Surely he needs nothing but his very own, this marvellous man who gave so lavishly to every one devoted to mathematics, or, indeed, to the highest advance of human thought in any form.

University of Texas.

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from March Number.]

III. PROOFS RESULTING FROM COMPARISON OF AREAS.

NOTE. Under this head, only a few varieties in connection with each type of figure will be given. The possible number of varieties of "dissection proofs" is absolutely unlimited.

XXXIII. Fig. 25.

Rectangle AM is equivalent to $2\triangle FAC$ is equivalent to $2\triangle EAB$ is equivalent to square EC. Similarly, rectangle BM is equivalent to square KC.

$\therefore$ Adding, square AH is equivalent to square $EC+$square KC.

Euclid's Proof. Prop. 47, Book I.

XXXIV. Fig. 25.

Rectangle AM is equivalent to parallelogram Aa=parallelogram AO is equivalent to square AD. Similarly, rectangle BM is equivalent to square BL.

$\therefore$ Adding, square AH is equivalent to square $AD+$square BL.

Edwards's Geometry, page 160.

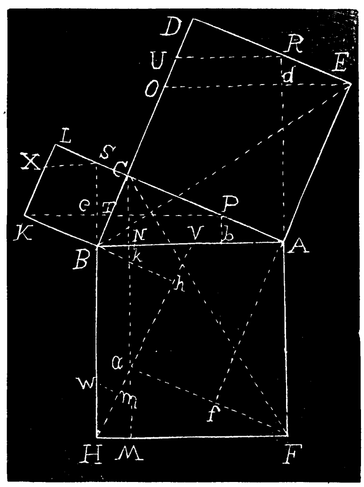


Fig. 25.

XXXV. Fig. 25.

$AB\cdot Ad$ is equivalent to $ABOE$ is equivalent to $ACDE$.

$AB\cdot Be$ is equivalent to $ABKP$ is equivalent to $BKLC$.

$\therefore$ Adding. $AB(Ad+Be)$ is equivalent to $AB(Ad+dR)$ is equivalent to $AB\cdot AF=ABHF$ is equivalent to $ACDE+BKLC$.

XXXVI. Fig. 25.

$AFMN$ is equivalent to $AFaC$ is equivalent to $AC \cdot Af = ACDE$. Similarly, $BHMN$ is equivalent to $BKLC$.

$\therefore$ Adding, $ABHF$ is equivalent to $ACDE + BKLC$.

Vieth, 1805.

XXXVII. Fig. 25.

$FMa = AdE$; $AfF = EDO$; $ANaf = AdOc$.

$\therefore$ $AFMN$ is equivalent to $ACDE$. Similarly, $BHMN$ is equivalent to $BKLC$.

$\therefore$ $ABHF$ is equivalent to $ACDE + BKLC$.

E. von Littrow, 1839.

XXXVIII. Fig. 25.

$AVaF = RUCA$; $FaH = AER$; $HhB = PLK$ is equivalent to $RDU + CLKT$; $BhV = KBT$.

$\therefore$ $ABHF$ is equivalent to $ACDE + BKLC$.

XXIX. Fig. 25.

$AVaF = RUCA$; $FaH = AER$; $RDU = HmW$; $BhmW = BKLS$; $BhV = BCS$.

$\therefore$ $ABHF$ is equivalent to $ACDE + BKLC$.

XL. Fig. 26.

$AFMR$ is equivalent to $ACNO$ is equivalent to $ACDE$. So, $BHMR$ is equivalent to $BKLC$.

$\therefore$ $ABHF$ is equivalent to $ACDE + BKLC$.

Sechhio, 1753.

XLI. Fig. 26.

$AFMR$ is equivalent to $AFaC = ESLD$ is equivalent to $ACDE$. So, $BHMR$ is equivalent to $BKLC$.

$\therefore$ $ABHF$ is equivalent to $ACDE + BKLC$.

Edwards's Geometry, page 158.

XLII. Fig 26.

$CAFU = ACNE$, and $CaU = CND$.

$\therefore$ $FaCA$ (is equivalent to $AFMR$) is equivalent to $ACDE$. So, $BHMR$ is equivalent to $BKLC$.

$\therefore$ $ABHF$ is equivalent to $ACDE + BKLC$.

XLIII. Fig. 26.

$Fa = AC = Va$.

$\therefore$ $Fa \cdot Va$ (is equivalent to $AFaC$) $= ACDE$.

$\therefore$ $ARMF$ is equivalent to $ACDE$. So, $BHMR$ is equivalent to $BKLC$.

$\therefore$ $ABHF$ is equivalent to $ACDE + BKLC$.

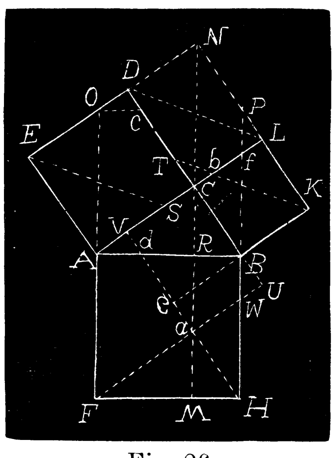


Fig. 26.

XLIV. Fig. 26.

$AFHd=OABc$ is equivalent to $AEOcC$.

$BeH=KBT$ is equivalent to $KBCb+ODc$. $Bed=KLb$.

$\therefore ABHF$ is equivalent to $ACDE+BKLC$.

XLV. Fig. 26.

$HaW=KLb$. $AFHaWB$ is equivalent to $ACBWF$ is equivalent to $ONPfA$ is equivalent to $ONCA+NPfC$ is equivalent to $ACDE+BKbC$.

$\therefore ABHF$ is equivalent to $ACDE+BKLC$.

To be Continued.]

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

Continued from April Number.]

PROPOSITION XXVII. *If a straight AX (Fig. 32.) drawn at any however small angle from the point A of AB, must at length meet (anyhow at an infinite distance) any perpendicular BX, which is supposed erected at any distance from this point A upon the secant AB: I say there will then be no more place for the hypothesis of acute angle.*

PROOF. From any point K chosen at will in AB near the point A, the perpendicular KL is erected to AB, which certainly (from Cor. II. of the preceding proposition) meets AX at a finite or terminated distance in some point L. But now it holds that there may be assumed in KB portions KK each equal to a certain assignable length R, and these more than any assignable finite number; since indeed the point B can be situated, in accordance with the present supposition, at however great a distance from this point A.

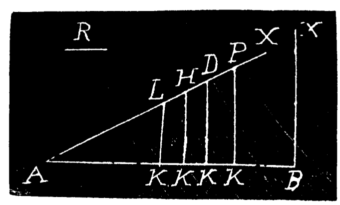


Fig. 32.

And accordingly from the other points K are erected to AB perpendiculars KH, KD, KP, which all (from the aforesaid corollary) meet the straight AX in certain points H, D, P; and so about the remaining points K uniformly designated toward the point B.

It holds secondly (from Eu. I. 16) that the angles at the points L, H, D, P will all be obtuse toward the parts of the points X; and just so (from Eu. I. 13) the angles at the aforesaid points will all be acute toward the point A.

Therefore (from Cor. II. after 3 of this) the side KH will be greater than the side KL ; the side KD greater than the side KH ; and so always proceeding towards the points X.

It holds thirdly that the four angles together of the quadrilateral $KLHK$ will be greater than the four angles together of the quadrilateral $KHDK$: for this in like case has already been demonstrated in XXIV of this.

It holds fourthly that the same is valid likewise of the quadrilateral $KHDK$ in relation to the quadrilateral $KDPK$; and so on always, proceeding to quadrilaterals more remote from this point A.

Since therefore are present (as in XXV of this) as many quadrilaterals described in the aforesaid mode, as there are, except the first LK, perpendiculars let fall from points of AX to the straight AB, it will hold uniformly (if we assume nine perpendiculars of this sort let fall, besides the first) the sum of all the angles which are comprehended by these nine quadrilaterals will exceed 35 right angles ; and therefore the four angles together of the first quadrilateral $KLHK$, which indeed in this regard has been shown the greatest of all, will fall short of four right angles by less than the ninth part of one right angle. Wherefore, these quadrilaterals being multiplied beyond any assignable finite number, proceeding always toward the parts of the points X, it holds in the same way (as in the same already recited theorem) that the four angles together of this stable quadrilateral $KHLK$ will fall short of four right angles less than any assignable little portion of one right angle.

Therefore these four angles together will be either equal to four right angles, or greater.

But then (from XVI of this) is established the hypothesis either of right angle or of obtuse angle ; and therefore (from V and VI of this) is destroyed the hypothesis of acute angle.

So then it holds, that there will be no place for the hypothesis of acute angle, if the straight AX drawn under however small angle from the point A of AB must at length meet (anyhow at an infinite distance) any perpendicular BX, which is supposed erected at any distance from this point A upon this secant AB.

Quod erat etc.

SOME DIVISIBILITY TESTS.

By WM. E. HEAL, Member of the London Mathematical Society, Marion, Indiana.

In the *Educational Times* for March, 1897, Professor Sylvester proposed the following problem : "If the digits r in number of any integer N read from left to right be multiplied repeatedly by the first r terms of the recurring series

1, 4, 3, -1, -4, -3 ; $\dot{1}$, $\dot{4}$, $\dot{3}$, $-\dot{1}$, $-\dot{4}$, $-\dot{3}$, show that, if the sum of these products be divisible by 13, so N will be, and not otherwise." The reason for the rule is apparent when we notice that 1, 4, 3, -1, -4, -3 are the remainders in reverse order of 10^1, 10^2, 10^3, 10^4, 10^5, 10^6 mod. 13 ; or what is the same thing in the development of $\frac{1}{13}$ as a circulating decimal.

Since we may prefix any number of ciphers to any number, it is clear that we may start with any number of the series only being careful to preserve the cyclical order. For example, we might equally as well write the series 3, -1, -4, -3, 1, 4.

Example. 11140640173 is divisible by 13 because $1(1)+4(1)+3(1)-1(4)-4(0)-3(6)+1(4)+4(0)+3(1)-1(7)-4(3)=-26=-2(13)$.

728 is divisible by (13) because $3(7)-1(2)-4(8)=-13$.

The reason for the rule suggests its extension to any number whatever.

Thus $\frac{1}{3}$ developed in a circulating decimal gives the constant remainder 1 and we have the well known rule that a number is divisible by 3 if the sum of its digits is so. $\frac{1}{7}$ developed in a circulating decimal gives the series 2, 3, 1, -2, -3, -1. Thus 6028620892 is divisible by 7 because $2(6)+3(0)+1(2)-2(8)-3(6)-1(2)+2(0)+3(8)+1(9)-2(2)=7$.

For 11 the remainders are 1, -1, and we have the known rule for divisibility by 11. For 13 the rule is as stated by Sylvester. For 17 we find the series 1, -5, 8, -6, -4, 3, 2, 7, -1, 5, -8, 6, 4, -3, -2, -7. Thus 442 is divisible by 17 because $3(4)+2(4)+7(2)=34=2(17)$.

For 19 we have the series 1, 2, 4, 8, -3, -6, 7, -5, 9, -1, -2, -4, -8, 3, 6, -7, 5, -9. It is clear that in this way we can find similar tests of divisibility for any number whatever, but it does not seem worth while to push the matter further except in special cases.

A simple rule for divisibility by 37 may be found in this way. The remainders are 1, -11, 10. Thus 343619 is divisible by 37 because $1(3)-11(4)+10(3)+1(6)-11(1)+10(9)=74=2(37)$.

May 7, 1897.

INTRODUCTION TO DIFFERENTIATION.

By JOHN MACNIE, A. M., Professor of Mathematics, University of North Dakota.

1. In the identity $\dfrac{r^n-1}{r-1}=r^{n-1}+r^{n-2}+\ldots\ldots r+1$, ...(1),

Since r may have any value, let $r=\dfrac{x^{1/m}}{z^{1/m}}$; then, by substituting this value for r in (1), multiplying both members by $z^{\frac{n-1}{m}}$, and simplifying, we obtain

$$\frac{x^{n/m}-z^{n/m}}{x^{1/m}-z^{1/m}}=x^{\frac{n-1}{m}}+x^{\frac{n-2}{m}}z^{\frac{1}{m}}+\;\ldots\ldots x^{\frac{1}{m}}z^{\frac{n-2}{m}}+z^{\frac{n-1}{m}}\;\ldots\ldots\ldots(2).$$

Dividing both members of (2) by the factor that rendered $x^{1/m}-z^{1/m}$ rational, we obtain, since, by (2). $x-z=(x^{1/m}-z^{1/m})(x^{\frac{n-1}{m}}+\ldots\ldots z^{\frac{n-1}{m}})$,

$$\frac{x^{n/m}-z^{n/m}}{x-z}=\frac{x^{\frac{n-1}{m}}+z^{\frac{n-2}{m}}z^{\frac{1}{m}}+\ldots\ldots x^{\frac{1}{m}}z^{\frac{n-2}{m}}+z^{\frac{n-1}{m}}}{x^{\frac{m-1}{m}}+x^{\frac{m-2}{m}}z^{\frac{1}{m}}+\ldots\ldots x^{\frac{1}{m}}z^{\frac{m-2}{m}}+z^{\frac{n-1}{m}}}\;\ldots\ldots\ldots(3),$$

which, as m may have any value, ± 1 included, is a general expression for the *ratio of the difference of two like powers to the difference of their bases.*

In (3), if we suppose $z=x$, since, then, there are in the numerator of the second member n terms, each$=x^{\frac{n-1}{m}}$, and in the denominator m terms, each$=x^{\frac{m-1}{m}}$ we obtain, for $z=x$,

$$\left[\frac{x^{\frac{n}{m}}-z^{\frac{n}{m}}}{x-z}\right]_{x=z}=\frac{0}{0}=\frac{nx^{\frac{n-1}{m}}}{mx^{\frac{m-1}{m}}}=\frac{n}{m}x^{(n/m)-1}$$

the first member assuming the indeterminate form on account of the presence in numerator and denominator of the factor $x^{1/m}-z^{1/m}$, which becomes zero by hypothesis. Hence, as m may have any value, the formula

$$\left[\frac{x^{n}-z^{n}}{x-z}\right]_{x=z}=nx^{n-1}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

holds true for every value of n. For the sake of simplicity of statement we shall suppose in what immediately follows $m=1$, and n=a positive integer.

Then (3) becomes

$$\frac{x^{n}-z^{n}}{x-z}=x^{n-1}+x^{n-2}z+\ldots\ldots xz^{n-2}+z^{n-1}\quad\ldots\ldots\ldots\ldots(3\ ).$$

2. Now, instead of regarding x and z in (3′) as unknown constants, we may regard them as denoting different values of the same variable z, as it varies from $z=0$, through $z=x$, toward $z=+\infty$. From this point of view we see that, assigning any two values to x and z, each member of (3′) expresses *the ratio of the increment of the power to the increment of the base, between these values;* or, briefly expressed, gives the *rate of increase of z^n.* For example, let $z=0$, $x=a$; then both members of (3′) become a^{n-1}, the *average* rate of increase of z^n while z increases from 0 to a; i. e. while z has increased by a units, z^n has increased a^{n-1} times as fast. We say "average rate" because, as will be seen by giving different values to a, the rate of increase of z^n is continually accel-

erating, just as the velocity or rate of motion of a falling body is continually accelerating.

3. If now we suppose $z=x-h$, h being infinitely small, the second member of (3′) will be *less* than nx^{n-1} by a difference infinitely small ; and if we suppose $z=x+h$, the second member of (3′) will be *greater* than nx^{n-1} by a difference infinitely small ; we infer, accordingly (the values of that second member being continuous) that nx^{n-1} represents the *rate of increase of z^n when z is passing through x.* For, if nx^{n-1} does not represent the rate of increase of z^n when z is passing through x, for what value of z does it represent the rate ?

The difficulty that is here experienced arises from the fact that we have here to deal, not with a constant ratio, as in algebra, but with a ratio that varies continuously as its terms vary, ratios of frequent occurence in physics and kindred sciences. Thus, when we say that a falling body at a certain point in its descent has a velocity of 50 feet a second, we do not mean that the body moves at that rate during any assignable period of time, but *would* descend that distance in a second, *if the motion continued uniform.* In the same way, nx^{n-1} does not mean the rate of increase of z^n during an interval of increase of z but the rate at which z^n would increase if the rate became constant from x.

From the limitation of our faculties, we are unable to realize the *absolute ;* as, for example, to draw or even conceive a straight line absolutely without breadth. Yet, while admitting this inability, we ignore in our reasonings upon straight lines all that is inconsistent with their definition. Similarly, while in our conception of a variable, a changing velocity for example, we can not help thinking of the *element* of change as constant for some interval, however minute, we here, again, ignore whatever is inconsistent with the definition of a variable as changing continuously. There is no objection, then, to our assisting our grasp of the idea by regarding a power of a variable as changing by infinitely small constant* *elements*, as long as we ignore inconsistent consequences.

4. Def. *Function*, as usual. Example, x^n a function of x.

5. Def. A variable being supposed to change by infinitely small *elements*, such an element is called the *differential* of the variable. The differential of a variable is denoted by the symbol d prefixed to the symbol of the variable. Thus dx, $d(x^n)$, are read respectively, *the differential of x, the differential of x^n.*

It has already been seen that $d(x^n)=nx^{n-1}dx$, that is when the variable is passing through the value x, the power is changing nx^{n-1} times as fast as the variable. Hence nx^{n-1} is called *the differential coefficient* of x^n, etc.

6. (Here would follow the demonstration of the rules for algebraic sums of variables, found much as usual. The rule for the differential of products may be found as follows, without the intervention of series.)

7. *To find the differential of the product of two variables,* say xy.

$\because\ 2xy=(x+y)^2-x^2-y^2.$

$\therefore\ 2d(xy)=2(x+y)\times d(x+y)-d(x^2)-d(y^2).$

*That is, constant during an infinitely small interval.

$$2d(xy)=2(x+y)(dx+dy)-2xdx-2ydy.$$

i. e. $d(xy)=xdx+xdy+ydx+ydy-xdx-ydy.$

i. e. $d(xy)=xdy+ydx.$

From this may be derived rules for $d(xyz)$, etc., and $d(x/y)$.

8. Here would follow demonstration by differentials of Binomial Formula for all values of n, with exercises.

9. Here would follow the algebraic deduction of some such formula as :

$$\log(1+z)=M(z-\tfrac{1}{2}z^2+\tfrac{1}{3}z^3-\ldots\ldots \text{ ad inf.})$$

$$\text{whence } d(\log 1+z)=M(1-z+z^2-\ldots\ldots\text{ad inf.})dx$$

$$d(\log 1+z)=M.\frac{1}{1+z}.dx$$

or, putting x for $1+z$ we have

$$d\log x=M(dx/x).$$

Whence may be derived $d(a^x)=a^x\log a$, etc.

10. Here would follow the algebraic deduction of

$$\sin x=x-(x^3/3\,!)+(x^5/5\,!)-(x^7/7\,!)+\ \ldots\ldots \qquad (1)$$

$$\text{and } \cos x=1-(x^2/2\,!)+(x^4/4\,!)-(x^6/6\,!)+\ldots\ldots \qquad (2).$$

From (1), $d(\sin x)=\{1-(x^2/2\,!)+(x^4/4\,!)-(x^6/6\,!)+\ldots\ldots\}dx=\cos x dx$,

and from (2), $d(\cos x)=-\sin x dx$, etc.

11. Then might follow applications to questions of maxima and minima, etc. Then deduction of Taylor's Theorem, with applications.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

79. Proposed by F. M. PRIEST, St. Louis, Mo.

How many \$20 gold pieces can be put in a room 20 feet long, 18 feet wide, 9 feet high?

Solution by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Spring-Springfield, Missouri.

A \$20 gold piece is about $\frac{8}{100}$ of an inch thick, and about $1\frac{7}{20}$ inches in diameter. By putting the pieces in cylindrical layers lengthwise of the room, we can place $(18\times12)\div1\frac{7}{20}$ or 160 cylinders in the first layer, each cylinder containing $(20\times12)\div\frac{8}{100}$ or 3000 \$20 gold pieces. By rectangular arrangement of the cylinders we can put in $(9\times12)\div1\frac{7}{20}$ or 80 layers. Hence, by this arrangement, we can put $80\times160\times3000=38,400,000$ pieces in the room.

By laying the cylinders of the second layer of cylinders between two cylinders of the first layer, the distance between the plane of centers of the first layer and the plane of centers of the second layer is $\sqrt{(\frac{27}{20})^2-(\frac{27}{40})^2}=\frac{27}{40}\sqrt{3}$. Hence, there can be placed in the room, by this arrangement, $(9\times12)\div\frac{27}{40}\sqrt{3}$ or 92 layers+.376 of a layer.

In these 92 layers 46 layers would contain 160 cylinders and 46 would cantain 159. But since there is still room at the top the last layer can be placed in so as to contain 160 cylinders.

Hence, there will be 47 layers of 160 cylinders and 45 layers of 159.

Since each cylinder contains 3000 \$20 gold pieces, there can be placed in the room by this method $(47\times160+45\times159)\times3000=44,025,000$ pieces.

It is possible that by considering other dimensions in the same way as the width in this solution a still larger number may be placed in the room.

Charles C. Cross obtained as his answer 38,400,000.

80. Proposed by CHARLES C. CROSS, Laytonsville, Maryland.

From a cask containing 10 gallons of wine, a servant drew off 1 gallon each day, for five days, each time supplying the deficiency by adding a gallon of water. Afterwards, fearing detection, he again drew off a gallon a day for five days, adding each time a gallon of wine. How many gallons of water still remained in the cask? [From *Quackenbos' Arithmetic*.]

Solution by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

Let 10 gallons$=a$, 1 gallon$=b$, the quantity of water or wine added after each draught, $\frac{1}{10}=b/a=1/n$, the part drawn off each time.

Then $a-a/n=a\left(\dfrac{n-1}{n}\right)$=quantity of wine left after first draught;

$a\left(\dfrac{n-1}{n}\right)-1/n$ of $a\left(\dfrac{n-1}{n}\right)=a\left(\dfrac{n-1}{n}\right)^2$=quantity of wine left after second draught;

$a\left(\dfrac{n-1}{n}\right)^2-1/n$ of $a\left(\dfrac{n-1}{n}\right)^2=a\left(\dfrac{n-1}{n}\right)^3$=quantity of wine left after third draught; and $a\left(\dfrac{n-1}{n}\right)^m$=quantity left after the mth draught$=A$.

Then $a-A$=water in the cask.

$A+b$=quantity of wine in cask before the $(m+1)$th draught since b gallons of wine are added.

$A+b-[(A/n)+(b/n)]+b=A\left(\frac{n-1}{n}\right)+b\left(\frac{n-1}{n}\right)$=quantity of wine before the $(m+2)$th draught.

$$A\left(\frac{n-1}{n}\right)+b\left(\frac{2n-1}{n}\right)-A\left(\frac{n-1}{n^2}\right)-b\left(\frac{2n-1}{n^2}\right)+b=A\left(\frac{n-1}{n}\right)^2-b\left(\frac{3n^2-3n+1}{n^2}\right)$$

=quantity of wine before the $(m+3)$th draught.

$$\therefore A\left(\frac{n-1}{n}\right)^p+b\left(pn^{p-1}-\frac{p(p-1)}{1.2}n^{p-2}+\ldots\ldots\right)+b$$

$$=A\left(\frac{n-1}{n}\right)^p+b\left(\frac{n^p-(n-1)^p}{n^p}\right)$$

=quantity of wine left after $(m+p)$th draught$=a\left(\frac{n-1}{n}\right)^{m+p}+b\left(\frac{n^p-(n-1)^p}{n^p}\right)$

In the present case, $a=10$, $b=1$, $1/m=\frac{1}{10}$, $m=5$, and $p=5$. Hence, substituting, we have $10\left[\frac{10-1}{10}\right]^{10}+1.\left[\frac{10^5-(10-1)^5}{10^5}\right]=7.581884401$ gallons, the quantity of wine left after putting in the last gallon of wine, and, therefore, 2.418115599 gallons=quantity of water in the cask.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

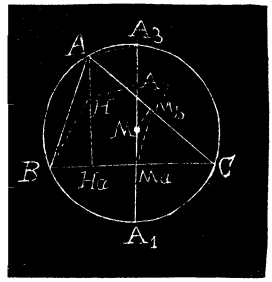
SOLUTIONS OF PROBLEMS.

71. Proposed by ROBERT J. ALEY, A. M., Ph. D., Professor of Mathematics, Indiana University, Bloomington, Indiana.

Prove by pure geometry: A perpendicular at the middle point, M_a, of the side BC of the triangle ABC meets the circumcircle in A'. On this perpendicular A'' and A''' are taken so that $M_aA''=M_aA'$ and $A''A'''=AH$. (H is the orthocenter of triangle ABC.) Prove that A''' is on the circumicircle.

Solution by F. M. McGAW, A. M., Professor of Mathematics, Bordentown Military Institute, Bordentown, New Jersey, and the PROPOSER.

Let $M_aA_1=M_aA_2$, $A_2A_3=AH$, to prove A_3 on the circumference of the circle. Since A_2A_3 is a line through M, the center of the circle, the proposition is in effect to prove A_3 one extremity of the diameter through M_a.



By the conditions $AH=A_2A_3$, and is parallel to it, therefore AHA_3A_2 is a parallelogram.

Also triangles BHA and M_aMM_b are similar, hence since $2M_aM_b=AB$, we have $AH=2MM_a$.

$$\begin{aligned}\text{Therefore, } A_1A_3&=A_2A_3+A_2M_a+M_aA_1\\&=AH+2M_aA_1\\&=2M_aM+2M_aA_1\\&=2(MA_1)=2r,\end{aligned}$$ hence A_3 is extremity of diameter.

Q. E. D.

Also solved by *CHAS. C. CROSS*, and *J. W. SCROGGS*.
Mr. Cross furnished two different solutions.

72. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

If a line with its extremities upon two curves move in any manner whatever, (the line may vary in length), and P a point upon the line which divides it in the ratio $m:n$ describe a curve, the area of this curve will be given by the formula—

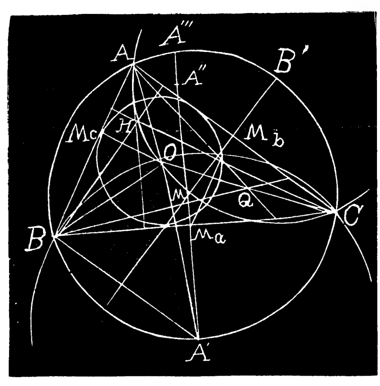
$$A=\frac{(m^2+nm)A_1+(n^2+mn)A_2-mnA_3}{(m+n)^2}.$$

No solution of this problem has been received.

73. Proposed by ROBERT J. ALEY, A. M., Ph. D., Professor of Mathematics, Indiana University, Bloomington, Indiana.

Prove by pure geometry: (1) A', B', and C' are the middle points of the arcs BC, CA, and AB respectively. With these points as centers, circles are described passing through B and C, C and A, and A and B respectively. Prove that these circles intersect in O, the center of the incircle of the triangle ABC; (2) that O, the center of the incircle, is Nagel's point of the triangle formed by joining the middle points of the sides.

Solution by CHARLES C. CROSS, Laytonsville, Maryland, and the PROPOSER.



(1) AO cuts the circumcircle at A', for AO bisects angle A and also its subtending arc. $\measuredangle OBA'=\frac{1}{2}(A+B)$.

$\measuredangle BOA'=\frac{1}{2}(A+B)$ for it is exterior angle to triangle BOA.

$\therefore$ triangle $A'BO$ is isosceles.

$A'B=A'O$. By similar reasoning it is proved that $B'A=B'O$ and $C'A=C'O$.

$\therefore$ The circles intersect in O.

(2) It is a well known property of Nagel's point that AQ and OM_a, BQ and OM_b, CQ and OM_c are respectively parallel.

The triangle $M_aM_bM_c$ is similar to the triangle ABC.

$$\measuredangle OM_aM_c = \measuredangle QAC.$$
$$\measuredangle OM_bM_c = \measuredangle QBC.$$
$$\measuredangle OM_cM_a = \measuredangle QCA.$$

$\therefore$ O with respect to the triangle $M_aM_bM_c$, is located precisely as Q is with respect to the triangle ABC.

Hence O is Nagel's point of triangle $M_aM_bM_c$.

Also solved by *F. M. McGAW* and *G. B. M. ZERR*.

74. Proposed by ROBERT J. ALEY, A. M., Ph. D., Professor of Mathematics, Indiana Universfty, Bloomington, Indiana.

Let O be the center of the inscribed circle. AO produced meets the circumcircle in A'. Find the ratio of AO to OA'.

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics, Ohio University, Athens, Ohio.

The coördinates of A are $\left(\frac{2\triangle}{a}, 0, 0\right)$; of O, (r, r, r); and of A', those of the intersection of $\beta-\gamma=0$......(1), with $a\beta\gamma+b\alpha\gamma+c\alpha\beta=0$......(2), having the constant relation $a\alpha+b\beta+c\gamma=2\triangle$......(3). These give for the coördinates

of A' $\left(-\frac{(b+c)^2}{a^2}-\frac{2\triangle}{a},\quad \frac{(b+c)^3}{a^3}+\frac{2\triangle(b+c)}{a^2},\quad \frac{(b+c)^3}{a^3}+\frac{2\triangle(b+c)}{a^2}\right)$.

The distance d between $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ is given by

$$d^2=-\frac{abc}{4\triangle^2}\{a(\beta_1-\beta_2)(\gamma_1-\gamma_2)+b(\gamma_1-\gamma_2)(\alpha_1-\alpha_2)+c(\alpha_1-\alpha_2)(\beta_1-\beta_2)\} \quad ...(4).$$

Putting $\alpha_1=(2\triangle/a)$, $\beta_1=0$, $\gamma_1=0$; $\alpha_2=\beta_2=\gamma_2=r$,

$$\overline{AO}^2=bcr(b+c-a)/2\triangle \quad .....(5).$$

Putting α_1, β_1, γ_1 equal respectively to the coördinates of A', and $\alpha_2=\beta_2=\gamma_2=r$ as before, in (4), we get an expression for $\overline{OA'}^2$

We can then express the ratio of OA to OA'.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

The point A' is evidently the middle point of arc BC. Since $\angle A'OC=\frac{1}{2}(A+C)$ and $\angle A'CO=\frac{1}{2}(A+B)$, $OA'=A'C=A'B$.

From Ptolemy's theorem, $ACA'B$ being a cyclic quadrilateral,

$$AB\times A'C+AC\times A'B=AA'\times BC, \text{ or}$$
$$c\times OA'+b\times OA'=(AO+OA')a.$$
$$\therefore\ OA : OA'=b+c-a : a=s-a : 2a.$$

Also solved by *G. B. M. ZERR* and *CHAS. C. CROSS*.

75. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, Ohio.

A plane passes through $(0,0,c)$ and touches the circle $x^2+y^2=a^2$, $z=0$; determine the locus of the ultimate intersections of the plane.

I. Solution by the PROPOSER.

Let the plane be $Ax+By+Cz+p=0$ ....(1).

Passing through $(0, 0, c)$, (1) gives $p=-cC$ ....(2),

and (1) becomes $Ax+By+Cz-cC=0$...... ....(3).

The x, y, z of (3) are those of $x^2+y^2=a^2$.........(4), $z=0$..... ...(5),

and also of $Ax+By-cC=0$...(6).

Making (4) homogeneous by aid of (6),

$$\left[\frac{1}{a^2}-\frac{A^2}{c^2C^2}\right]\frac{x^2}{y^2}-\frac{2AB}{c^2C^2}x/y+\left[\frac{1}{a^2}-\frac{B^2}{c^2C^2}\right]=0 \quad ......(7).$$

For (3) to touch (7), the values of x/y from (7) must be equal, or

$$\left[\frac{1}{a^2}-\frac{A^2}{c^2C^2}\right]\left[\frac{1}{a^2}-\frac{B^2}{c^2C^2}\right]=\frac{A^2B^2}{c^4C^4}\ ....(8),$$

$$\text{or,}\quad A^2/C^2+B^2/C^2-c^2/a^2=0\ ....(9).$$

From (3), $A/C=(z-c)/x-(y/x)(B/C)$...........................(10).

Substituting (10) in (9), etc.,

$$\frac{x^2+y^2}{x^2}B^2/C^2-\frac{2y(z-c)}{x^2}B/C+\frac{(z-c)^2}{x^2}-c^2/a^2=0\ ....(11),$$

a quadratic in the undetermined constant B/C, giving the envelope

$$\frac{x^2+y^2}{a^2}=\frac{(z-c)^2}{c^2}\ ....\ .....(12).$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Let the plane touch the circle at the point (x', y').

$\therefore \dfrac{xx'}{a^2}+\dfrac{yy'}{b^2}+z/c=1$, is the equation to the plane, but

$x'^2+y'^2=a^2$..............(1). $\therefore dy'/dx'=-x/y=-x'/y'$...............(2).

(2) in the equation to the plane gives,

$$x'=\frac{a^2x(c-z)}{c(x^2+y^2)},\quad y'=\frac{a^2y(c-z)}{c(x^2+y^2)}$$

These values of x', y' in (1), $x^2+y^2-(a^2/c^2)(c-z)^2=0$, a cone of revolution as the locus.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

49. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

A rectangular stick of timber of known dimensions is placed upon a platform of given height in a vertical position with the center above the edge of platform, and slightly displaced from the vertical. Where and in what manner will it strike the ground?

I. Solution by the PROPOSER.

Any body rotating about the center of an end has the energy,

$$\tfrac{1}{3}(\omega^2)m(a^2+b^2).$$

If the body has fallen through the angle θ, the energy is

$$\tfrac{1}{2}b(1-\cos\theta)m.$$

$$\therefore\ 2\omega^2(a^2+b^2)=3b(1-\cos\theta)+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The body will leave the platform when the statical pressure=centrifugal force. The pressure=$m\cos\theta$. Centrifugal force=$\frac{1}{2}m\omega^2b$.

$$\therefore\ \omega^2b=2\cos\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) and (2), $\omega=\sqrt{\dfrac{6b}{4a^2+7b^2}}$ .... (3), and $\cos\theta=\dfrac{3b^2}{4a^2+7b^2}$ ....(4).

Take the edge of platform as origin. Let the axis of x be horizontal and the axis of y vertical. Resolve the angular velocity of the center of gravity into its vertical and horizontal components at the instant of the stick leaving the platform.

$$\left.\begin{array}{l} V_x=\frac{1}{2}b\omega\cos\theta \\ V_y=\frac{1}{2}b\omega\sin\theta \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

For the accelerations we have $\dfrac{d^2x}{dt^2}=0, \qquad \dfrac{d^2y}{dt^2}=-g.$

Then $\dfrac{dx}{dt}=c$, and $\dfrac{dy}{dt}=-gt+c_2$.

Let us begin to reckon time from the instant that the body leaves the platform.

Then $\dfrac{dx}{dt}=V_x$, and $\dfrac{dy}{dt}=-gt+V_y$.

$x=V_xt+c_3$. $y=-\frac{1}{2}gt^2+V_yt+c_4$.

When $t=0$, $x=\frac{1}{2}b\sin\theta$, and $y=\frac{1}{2}b\cos\theta$.

$$\left.\begin{array}{l}\text{Then } x=V_xt+\frac{1}{2}b\sin\theta \\ \text{and } y=-\frac{1}{2}gt^2+V_yt+\frac{1}{2}b\cos\theta\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ (6).$$

These give the motion of the center of gravity.

Call T the time taken for one end to reach the ground. Then after leaving the platform it will have rotated through the angle $T\omega$.

It therefore makes an angle $\theta+T\omega$ with the vertical.

The center of gravity has fallen,

$$Y_1=-\tfrac{1}{2}gT^2+V_yT+\tfrac{1}{2}b\cos\theta. \quad \text{Also } X_1=V_xT+\tfrac{1}{2}b\sin\theta.$$

The center of gravity will be the distance $\frac{1}{2}b\cos(\theta+T\omega)$ from the ground.

Now $Y_1+\frac{1}{2}b\cos(\theta+T\omega)=H$, the height of tower.

Or, $\frac{1}{2}b\cos(\theta+T\omega)-\frac{1}{2}gT^2+V_yT+\frac{1}{2}b\cos\theta=H$ ....(7).

From this equation T may be determined. The horizontal distance from the foot of the tower will then be given by the equation,

$$X=X_1+\tfrac{1}{2}b\cos(\theta+T\omega)=V_xT+\tfrac{1}{2}b\sin\theta+\tfrac{1}{2}b\sin(\theta+T\omega).$$

II. Solution by **ALFRED HUME, C. E., D. Sc.**, Professor of Mathematics, University of Mississippi, P. O. University, Mississippi.

The stick turns about its lower extremity until it reaches a horizontal position with an angular velocity given by $\omega^2=(3g/2a)$, $2a$ being its length.

Subsequently there are two motions which may be considered independently. One is that of rotation about the center of gravity with a constant angular velocity, ω; the other, that of translation, the center of gravity falling vertically with an initial velocity, $a\omega$.

Estimating the motion from the horizontal position, the stick is vertical when it has turned through an odd number of right angles; that is, at the end of $n\pi/2\omega$ seconds, n being any odd number. If S_v denote the distance from the level of the platform to the lowest point of the stick at the instants of verticality, the motion of translation gives,

$$S_v - a = a\omega\frac{n\pi}{2\omega} + \tfrac{1}{2}g\left(\frac{n\pi}{2\omega}\right)^2,$$

or, substituting the value of ω,

$$S_v = \{1 + \tfrac{1}{2}(\pi n)[1 + \tfrac{1}{8}(\pi n)]\}a, \ n \text{ being odd.}$$

Similarly the positions of horizontality are given by

$$S_h = \tfrac{1}{2}(\pi n)[1 + \tfrac{1}{8}(\pi n)]a, \ n \text{ being even.}$$

If any value of S_v or S_h equals D, the distance from the platform to the ground, the stick will strike the ground, in the one case vertically, in the other, horizontally.

The discussion might be continued in general terms. Instead of this, however, let $a=1$ foot, and $D=10$ feet.

Giving to n the values of 1, 2, and 3 in the proper equations, the first and second values of S_v are found to about 3.4 and 13.1, and the first value of S_h about 6.4. Consequently the stick will strike the ground in passing from a horizontal toward a vertical position.

Since in falling 6.4 feet a half revolution has been made, the time for this motion is π/ω seconds, and the velocity of the center of gravity when the stick is horizontal for the last time is $\omega + g(\pi/\omega)$, remembering that $a=1$. If the stick turns through an angle θ before striking the ground, the center of gravity falls through $(3.6 - \sin\theta)$ feet in θ/ω seconds, giving the equation,

$$3.6 - \sin\theta = [\omega = g(\pi/\omega)](\theta/\omega) + \tfrac{1}{2}g[(\theta/\omega)]^2,$$

which reduces to $3.6 - \sin\theta = 3.1\theta + \tfrac{1}{3}\theta^2$, approximately ; from which $\theta = 48^\circ\ 20'$, about.

The horizontal distance from the edge of the platform to the point at which the stick touches the ground is $1 + \cos\theta$, or 1 foot, 8 inches, approximately.

θ is, of course, the inclination of the stick to the horizontal at the instant of contact with the ground.

In the last part of this work the thickness of the stick has been neglected.

50. Proposed by SYLVESTER ROBINS, North Branch Depot, New Jersey.

The edges of a rectangular parallelopiped are within 1 of the proportion $2:3:9$, and they are $2x\pm1$, $3x$ and $9x$, $(2x\mp1)^2+(3x)^2+(9x)^2=$the diagonal squared$=94x^2\mp4x+1=\square$. To find four integral values for x.

I. Solution by A. H. HOLMES, Box 963, Brunswick, Maine.

We may put it in the form : $90x^2+(2x\pm1)^2=\square$, or

$$m^2x^2-(m^2-90)x^2+(2x\pm1)^2=\square.$$

$\therefore\ 2m(2x\pm1)=(m^2-90)x$; $4mx\pm2m=m^2x-90x$.

$\therefore\ x=\pm(2m/(m^2-4m-90)$.

Let $m=nx$. Then $n^2x^2-4nx-90=\pm2n$; $n^2x^2-4nx+4=94\pm2n$.

Take plus sign and let $n=3$. $\therefore\ 3x=2+10=12$. $\therefore\ x=4$.

Now let $n=a/b^2$. $a^2x^2/b^4-4ax/b^2+4=94\pm2a/b^2=(94b^2\pm2a)/b^2$

Now take $b=3$. $\therefore\ a=5/2$ and $a/b^2=5/18$.

$\therefore\ 5x/18=2+29/3$. $5x=36+174=210$. $\therefore\ x=42$.

Now let $b=10$. $\therefore\ a=9/2$ and $a/b^2=9/200$.

$\therefore\ 9x/200=2+97/10$. $9x=400+1940=2340$. $\therefore\ x=260$.

Now let $b=23$. $\therefore\ a=-3/2$ and $a/b^2=--3/1058$.

$\therefore\ -3x/1058=2-223/23$, or $3x=8142$. $\therefore\ x=2714$.

For $x=$ 4 we have : $94x^2+4x+1=\square$.

For $x=$ 42 we have : $94x^2-4x+1=\square$.

For $x=$ 260 we have : $94x^2+4x+1=\square$.

For $x=2714$ we have : $94x^2-4x+1=\square$.

II. Solution by A. H. BELL, Hillsboro, Illinois.

The equation readily reduces to : $t^2-94y^2=-90$..................(1), and $x=(t\mp2)/94$ ....(2). $(1)\div9$ gives $t'^2-94y'^2=-10$, and $t=3t'$, $y=3y'$..................(3).

One cycle.

$\sqrt{94}$. No. of Frac's:		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16:
Quotients complete Denom'rs	1 :	13	6	5	9	10	3	15	2	15	3	10	9	5	6	13	1:
Quotients	9 :	1,	2,	3,	1,	1,	5,	1,	(8),	1,	5,	1,	1	3	2	1	18:
Convergents	$\frac{1}{0}$, $\frac{9}{1}$:	$\frac{10}{1}$,	$\frac{29}{3}$,	$\frac{97}{10}$,	$\frac{126}{13}$,	$\frac{223}{23}$,	$\frac{1241}{128}$,	$\frac{1464}{151}$,	$\frac{12953}{1336}$,	$\frac{14417}{1487}$,	$\frac{85038}{8771}$,	$\frac{99455}{10258}$,	$\frac{184499}{19029}$,	$\frac{652934}{67345}$,	$\frac{1490361}{155719}$,	$\frac{2143295}{221064}$.—	

The convergents preceding the denominators, 10 of the complete quotients $=t'/y'=126/13$ and $85038/8771$ ....(4), as they are even fractions.

$\therefore$ answer the -10 of (3). To obtain other values of t' and y', take $v^2-94u^2=1$..................(5).

$$\left.\begin{array}{r}(3)\times(5)\text{ and }\pm188t'uvy'\quad(t'v\pm94uy')^2-94(t'u\pm vy')^2=-10\\ \text{or, }t_n'^2-94y_n'^2=-10\end{array}\right\}\ldots\ldots\ldots\ldots\text{(6)}.$$

The smallest integral values for $v/u=2143295/221064$, but as fractional

values for t' and y' can be used as shown in (3), to obtain these we solve (5).

Let $v=v'/z$ and $u=u'/z$; then $v'^2-z^2=94u'^2$ ....(7).

Now let $u^2=pq$ and let $94=$any two factors, then (7) can be made

$$\left.\begin{array}{l} v'+z=p^2 \text{ or } 2p^2 \\ v'-z=94q^2 \text{ or } 47q^2 \end{array}\right\}$$

add and subtract, etc. $v'=p^2+94q^2$ or $2p^2+47q^2$; $z=p^2-97q^2$ or $2p^2-47q^2$; $u'=2pq$..................(8).

In the right-hand values if $p=5$ and $q=1$, $v'=97$; $z=3$; $u'=10$. There are an infinite number of values but these are the only ones admissible.

(7) $v=97/3$ and $u=10/3$; substituting these along with those of (4) separately in (6) we have $t_n'=2/3$ and $24442/3$; and $t_n'=3946/3$ and $16493426/3$ with those in (4), will make six values for t', and now in (3) and (2) $x=0$, 4, -42, 260, -2714, and 175462, etc. The sign=side($2x\pm1$). $y=94$, 39, 407, 2521, 26313.

III. Solution by the PROPOSER.

This problem is suggested by a remark in No. 5, Vol. I.: "$x^2-94y^2=\pm1$; this is the most difficult number under 100."

1. Find initial terms in that infinite series of rational rectangular solids where the edges of each term are in proportion as 2 : 3 : 9, within 1 in the *thickness*.

Let $2x\pm1$, $3x$ and $9x$ be the edges ; then $94x^2\pm4x+1=\square=(mx\pm1)^2=m^2x^2\pm2mx+1$. $x=(\pm2m\mp4)/(94-m^2)$.

Say $m=\sqrt{(94)}=9/1$, $10/1$, $29/3$, $97/10$, $126/13$, $223/23$, $1241/128$, $1464/151$, etc.

When $m=$	10	29/3	97/10	223/2x	
Then $x=$	4	42	260	2714	
$2x\pm1=$	9	83	521	5427	Thickness.
$3x=$	12	126	780	8142	Width.
$9x=$	36	378	2340	24426	Length.
$\sqrt{(94x^2\pm4x+1)}=$	39	407	2521	26313	Solid diagonal.

2. Find first term in an infinite series of rational parallelopipeds where the dimensions of every solid are in proportion as 2 : 3 : 9, within 1 in the *width*.

Let $2x$, $3x\pm1$ and $9x$ represent the edges. Then $94x^2\pm6x+1=\square=(mx\pm1)^2=m^2x^2\pm2mx+1$. Whence $x=(2m\mp6)/(94-m^2)$, $m=\sqrt{(94)}=9$, 10, 29/3, 97/10, 126/13, etc.

$m=$	29/3	126/13
$x=$	24	429
$2x=$	48	858
$3x\pm1=$	73	1286
$9x=$	216	3861
Solid diagonal=	233	4159

3. Find a term in an infinite series of rational parallelopipeds where the edges are in proportion as 2 : 3 : 9, within unity in *length*.

Let $2x$, $3x$, and $9x\pm1$ be the edges. $94x^2\pm18x+1=\square=(mx\pm1)^2=m^2x^2\pm2mx+1$. $x=(2m\mp18)/(94-m^2)$ Substitute $m=1464/151$, and $x=15855$, $2x=31710$, $3x=47565$, $9x-1=142694$.

Proof: $31710^2+47565^2+142694^2=153719^2$.

4. Find some term in an infinite series of rational parallelopipeds where the dimensions come within 1 unit in the *thickness* of being in proportion as 3 : 6 : 7.

Let edges be $3x\pm1$, $6x$ and $7x$. $94x^2\pm6x+1=\square=(mx\pm1)^2=m^2x^2\pm2mx+1$. $x=(2m\mp6)/(94-m^2)$.

When $m=29/3$		$m=126/33$	
$x=$	24	$x=$	429
$3x\pm1=$	144	$3x\pm1=$	1286 etc.
$6x=$	144	$6x=$	2574
$7x=$	168	$7x=$	3003
S. d.$=$	233	S. d.$=$	4159

Proof: $73^2+144^2+168^2=233^2$

5. Find some term in an infinite series of rational rectangular solids where the edges come within 1 unit in the *width* of being in the proportion of 3 : 6 : 7. Let the edges be represented by $3x$, $6x\pm1$ and $7x$. Then $94x^2\pm12x+1=\square=(mx\pm1)^2=m^2x^2+2mx+1$. $x=(2m\mp12)/(94-m^2)$. When $m=\sqrt{94}$ ......1464/151. Then $x=84258$ or 357870.

$3x=252774$	or $3x=1073610$
$6x-1=505547$	$6x+1=2147221$
$7x=589806$	$7x=2505090$
Diagonal$=816911$	Diagonal$=3469679$

6. Find a term in that infinite series of rational parallelopipeds wherein the edges of every solid are within unity in the *length* of being in proportion to each other as 3 : 6 : 7.

$(3x)^2+(6x)^2+(7x\pm1)^2=94x^2\pm14x+1=\square=(mx\pm1)^2$

$94x\pm14=m^2x\pm2m$. $x=(2m\mp14)/(94-m^2)$. $m=\sqrt{94}$. Now when $m=29/3$, $x=60$, $3x=180$, $6x=360$, $7x-1=419$.

$180^2+360^2+419^2=581^2$.

Also solved by *J. H. DRUMMOND.*

51. Proposed by H. C. WILKES, Skull Run, West Virginia.

The difference between the roots of two successive triangular square numbers, [i. e. triangular numbers that are also square numbers], equals the sum of two successive integral numbers, the sum of whose squares will be a square number. Demonstrate. Or, if s and t be the roots of any two successive triangular number that are also square numbers, prove that $t-s=2n+1$, where $n^2\ (n+1)^2=\square$.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

$$\frac{n(n+1)}{2}\text{ is a square when }n=\frac{(1+\sqrt{2})^{2m}+(1-\sqrt{2})^{2m}-2}{4}$$

$$\therefore \pm\sqrt{\frac{n(n+1)}{2}} = \pm\left\{\frac{(1+\sqrt{2})^{2m}-(1-\sqrt{2})^{2m}}{4\sqrt{2}}\right\} \quad \ldots\ldots\ldots\ldots\ldots(1).$$

$$\pm\sqrt{\frac{n'(n'+1)}{2}} = \pm\left\{\frac{(1+\sqrt{2})^{2m+2}-(1-\sqrt{2})^{2m+2}}{4\sqrt{2}}\right\} \quad \ldots\ldots\ldots\ldots\ldots(2).$$

Taking (2)+ and (1)−, and then taking their difference, we easily get,

$$\frac{(1+\sqrt{2})^{2m+2}-(1-\sqrt{2})^{2m+2}}{4\sqrt{2}}+\frac{(1+\sqrt{2})^{2m}-(1-\sqrt{2})^{2m}}{4\sqrt{2}}=2y+1.$$

$$\therefore \frac{(1+\sqrt{2})^{2m+1}+(1-\sqrt{2})^{2m+1}}{2}=2y+1.$$

$$\therefore \left\{\frac{(1+\sqrt{2})^{2m+1}+(1-\sqrt{2})^{2m+1}}{4}-\tfrac{1}{2}\right\}^2+$$

$$\left\{\frac{(1+\sqrt{2})^{2m+1}+(1-\sqrt{2})^{2m-1}}{4}+\tfrac{1}{2}\right\}^2=y^2+(y+1)^2.$$

$$\therefore 2\left\{\frac{(1+\sqrt{2})^{2m+1}+(1-\sqrt{2})^{2m+1}}{4}\right\}^2+\tfrac{1}{2}=y^2+(y+1)^2.$$

$$\therefore \left\{\frac{(1+\sqrt{2})^{2m+1}-(1-\sqrt{2})^{2m+1}}{2\sqrt{2}}\right\}^2=y^2+(1y+1)^2.$$

In above m can have any positive integral value.

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

This problem is true if we read "The sum of" instead of "The difference between." It might also be stated as follows: The difference between the roots of two successive triangular square numbers equals a number whose square is the sum of the squares of two successive integral numbers.

From Solution III of Problem 36, Vol. III., No. 3, page 82, we find that when one of the triangular square numbers is $n(n+1)/2$, the next in order, in terms of n, is $\left(2n+1+3\sqrt{\frac{n(n+1)}{2}}\right)^2$

The *difference* of the two roots is $2n+1+2\sqrt{\frac{n(n+1)}{2}}$.

The *sum* of the two roots is $2n+1+4\sqrt{\frac{n(n+1)}{2}}$, which equals the sum of the two consecutive integral numbers, $n+2\sqrt{\frac{n(n+1)}{2}}$ and $n+1+2\sqrt{\frac{n(n+1)}{2}}$.

But $\left(n+2\sqrt{\frac{n(n+1)}{2}}\right)^2+\left(n+1+2\sqrt{\frac{n(n+1)}{2}}\right)^2=6n^2+6n+1+(8n+4)\sqrt{\frac{n(n+1)}{2}}$

which equals the square of the *difference* of the two roots, or

$$\left(2n+1+2\sqrt{\frac{n(n+1)}{2}}\right)^2.$$

Illustration.—From the series of triangular square numbers, 1^2, 6^2, 35^2, 204^2, 1189^2, etc., take 6 and 35. $35-6=29$; $35+6=41=20+21$; $20^2+21^2=29^2$.

This problem and problems No. 45, (Vol. III., No. 5, page 153), and No. 36, of Diophantine Analysis, are very closely related.

Also solved by the *PROPOSER*.

52. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in Columbiam University, Washington, D. C.

Prove that a "magic square" of nine integral elements, whose rows, columns, and diagonals have a constant sum, is only possible when this sum is a multiple of three.

I. Solution by M. W. HASKELL, M. A., Ph. D., Associate Professor of Mathematics, University of California, Berkeley, California.

Let the magic square be $\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & f \\ \hline g & h & k \\ \hline\end{array}$ and let S be the constant sum.

Then $S=a+b+c=d+e+f=g+h+k=a+d+g=b+e+h=c+f+k=a+e+k=c+e+g$.

Adding these all together, we have $8S=3a+2b+3c+2d+4e+2f+3g+2h+3k=3(a+c+g+k)+2(b+e+h)+2(d+e+f)$. But the last two quantities in parenthesis are each$=S$. Hence $4S=3(a+c+g+k)$, and S is a multiple of 3.

II. Solution by —— (Paper Unsigned.)

Suppose the numbers occupying the magic square to be a, b, c, d, e, f, g, h, k. Now $a+e+k=b+e+h=c+e+g=S$.

$\therefore$ $a+k\equiv k(\text{mod } 3)$, $b+h\equiv k(\text{mod } 3)$, $c+g\equiv k(\text{mod } 3)$, where $S-e\equiv k(\text{mod } 3)$.

Adding the congruences, $(a+b+c)+(g+h+k)\equiv 0(\text{mod } 3)$. Or, since $(a+b+c)+(g+h+k)\equiv 0(\text{mod } 3)$, $2S\equiv 0(\text{mod } 3)$.

Multiply by 2, and divide by 3, and the result is $S\equiv 0$. Q. E. D.

III. Solution by W. H. CARTER, Professor of Mathematics, Centenary College of Louisiana, Jackson, Louisiana.

Let the rows of the "square" be a, b, c ; x, y, z ; and l, m, n, and let the constant sum be k. We have to show that $k/3$ is integral. We have $a+y+n=k$; $b+y+m=k$; $l+y+c=k$. Add, and we have $(a+b+c)+(l+m+n)+3y=3k$, that is, $2k+3y=3k$.

$\therefore$ $3y=k$. $\therefore$ $y=k/3$. But y is integral. $\therefore$ $k/3$ is integral.

Also solved by *M. A. GRUBER* and *G. B. M. ZERR*.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

48. Proposed by P. H. PHILBRICK, C. E., Pineville, Louisiana.

A, B, C, D, and E play with dice, each throwing three, three successive times, for a stake a. A, B, and C throw; C throwing the highest, 52. What is his expectation?

I. Solution by the PROPOSER.

If D or E or both throw 52, C gets but a part of the stake. If D or E or both throw 53 or 54, C gets none of the stake.

$52=18+18+16=18+17+17$. $53=18+18+17$. $54=18+18+18$.

The chance of throwing 16 at a single throw is $\frac{6}{216}$.

The chance of throwing 17 at a single throw is $\frac{3}{216}$.

The chance of throwing 18 at a single throw is $\frac{1}{216}$.

Hence since D may throw 16 (or 18) at any one of the three throws, his chance of throwing 52 at three throws is $3(\frac{1}{216}\times\frac{1}{216}\times\frac{6}{216})+3(\frac{1}{216}\times\frac{1}{216}\times\frac{3}{216})=p_1$ say. E has the same chance of reaching the same result. The chance that D (or E) will not throw 52 is $(1-p_1)$; and the chance that D or E will throw 52 and the others not is $p_1(1-p_1)$, in which case the expectation is $p_1(1-p_1)\frac{1}{2}a$.

The chance that D, and E also, will throw 52 is p_1^2, in which case their joint expectation is $p_1^2\frac{2}{3}a$. Hence the expectation of D or E or of both, coming from throwing 52 is, $2p_1(1-p_1)\frac{1}{2}a+p_1^2\frac{2}{3}a=p_1(3-p_1)\frac{1}{3}a$.

The chance of D or E throwing 53 is, $3(\frac{1}{216}\times\frac{1}{216}\times\frac{3}{216})=p_2$; and the chance that one or both will throw 53 is, $2p_2(1-p_2)+p_2^2=p_2(2-p_2$; and their joint expectation is, $p_2(2-p_2)a$.

The chance that D or E will throw 54 is $(\frac{1}{216}+\frac{1}{216}+\frac{1}{216})=p_3$); and the chance that one or both will throw 54 is, $2p_3(1-p_3)+p_3^2=p_3(2-p_3)$; and their joint expectation is, $p_3(2-p_3)a$. Hence C's expectation is,

$$\{1-\tfrac{1}{3}[p_1(3-p_1)]-p_2(2-p_2)-p_3(2-p_3)\}a=(1+325p_3^2-47p_3)a.$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas, and J. SCHEFFER, A. M., Hagerstown, Maryland.

D and E may each throw 52, 53, or 54.

52 can be thrown as follows: (6, 6, 6), (6, 6, 5), (6, 6, 5); (6, 6, 6), (6, 6, 6), (6, 6, 4).

53 can be thrown as follows: (6, 6, 6), (6, 6, 6), (6, 6, 5).

54 can be thrown as follows: (6, 6, 6), (6, 6, 6), (6, 6, 6).

D's chance of throwing 52, 53, or 54 is,

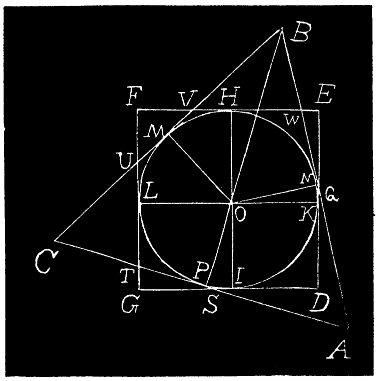
$$p=\frac{9}{(216)^3}+\frac{3}{216^3}+\frac{3}{216^3}+\frac{1}{216^3}=\frac{16}{(216)^3}=\frac{2^4}{6^9}.$$

$1-p=1-2^4/6^9=\frac{10077680}{10077696}=P$, chance that D will throw less than 52.
P^2=chance that D and E will both throw less.
$\therefore\ aP^2=C$'s expectation on the supposition that C wins and no ties.

49. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

A square whose side is $2a$ and an equilateral triangle whose altitude is $3a$ are fastened together at their centers, but otherwise free to move. If they are thrown on a floor at random, what is the average area common to both ?

Solution by HENRY HEATON, M. Sc., Atlantic, Iowa, and the PROPOSER.



In the figure let O be the common center of the square and the triangle. Then $OK=ON=OH=OM=OL=OP=OI=a$.

Let the triangle $KON=2\theta$.

Then $\angle NOH=\frac{1}{2}\pi-2\theta$, $\angle HOM=\frac{2}{3}\pi-(\frac{1}{2}\pi-2\theta)=\frac{1}{6}\pi+2\theta$, $\angle MOL=\frac{1}{3}\pi-(\frac{1}{6}\pi+2\theta)=\frac{1}{6}\pi-2\theta$, $\angle LOP=\frac{2}{3}\pi-(\frac{1}{6}\pi-2\theta)=\frac{1}{2}\pi+2\theta$, and $\angle POI=\frac{1}{2}\pi-(\frac{1}{3}\pi+2\theta)=\frac{1}{6}\pi-2\theta$.

Area of surface, $KONQ$, $=a^2\tan\theta$;
area of surface, $NOHW$, $=a^2\tan(\frac{1}{4}\pi-\theta)$;
area of surface, $HOMV$, $=a^2\tan(\frac{1}{12}\pi+\theta)$;
area of surface, $MOLU$, $=a^2\tan(\frac{1}{6}\pi-\theta)$;
area of surface, $KOPT$, $=a^2\tan(\frac{1}{6}\pi+\theta)$;
area of surface, $POIS$, $=a^2\tan(\frac{1}{12}\pi-\theta)$;
area of square, $OKDI$, $=a^2$.

Hence the area common to the square and triangle is

$$S=a^2[1+\tan\theta+\tan(\tfrac{1}{4}\pi-\theta)+\tan(\tfrac{1}{12}\pi+\theta)+\tan(\tfrac{1}{6}\pi-\theta)+\tan(\tfrac{1}{6}\pi+\theta)+\tan(\tfrac{1}{12}\pi-\theta)].$$

The positions for $\theta>\frac{1}{12}\pi$ are exact repetitions of those for $\theta<\frac{1}{12}\pi$.
Hence the required average area is

$$\Delta=\int_0^{\frac{1}{12}\pi}Sd\theta\div\int_0^{\frac{1}{12}\pi}d\theta=\frac{12a^2}{\pi}\int_0^{\frac{1}{12}\pi}\Big[1+\tan\theta+\tan(\tfrac{1}{4}\pi-\theta)+\tan(\tfrac{1}{12}\pi+\theta)+$$

$$\tan(\tfrac{1}{6}\pi-\theta)+\tan(\tfrac{1}{6}\pi+\theta)+\tan(\tfrac{1}{12}\pi-\theta)\Big]d\theta=a^2\Big[1+\frac{12}{\pi}\log_e2\Big].$$

This problem was also solved in a very excellent manner by G. B. M. Zerr.

50. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Find (1), the average length of all straight lines having a given direction, between 0 and a; (2), the average length of chords drawn from one extremity of the diameter a of a semi-circle to all points in the semi-circumference; and (3), find the average area of all triangles formed by a straight line of constant length a sliding between two straight lines at right angles.

Solution by HENRY HEATON, M. Sc., Atlantic, Iowa.

(1). Let x=length of one of the straight lines. Then the required average is $\int_0^a xdx \div \int_0^a dx = \frac{1}{2}a$.

(2). Let θ=angle between the chord and diameter. Then the length of the chord is $2a\cos\theta$, and the average length of all chords is

$$2a\int_0^{\frac{1}{2}\pi}\cos\theta d\theta \div \int_0^{\frac{1}{2}\pi} d\theta = 4a/\pi.$$

(3). Let θ=the angle between the sliding line and one of the fixed ones. The area of the triangle is $(\frac{1}{2}a^2)\sin\theta\cos\theta$. The average area of all such triangles depends upon circumstances. If the areas be taken at equal angular intervals, the required average is $A_1 = \int_0^{\frac{1}{2}\pi}(\frac{1}{2}a^2)\sin\theta\cos\theta d\theta \div \int_0^{\frac{1}{2}\pi} d\theta = a^2/2\pi$.

If the areas be taken at equal intervals as measured on one of the fixed lines along which the end of the sliding line moves, the average area is

$$A_2 = \int_0^{\frac{1}{2}\pi}(\tfrac{1}{2}a^2)\sin\theta\cos\theta d(a\sin\theta) \div \int_0^{\frac{1}{2}\pi} d(a\sin\theta) = \tfrac{1}{2}a^2\int_0^{\frac{1}{2}\pi}\sin\theta\cos\theta d\theta = \tfrac{1}{6}a^2.$$

This is but a repetiton of Problem 26, about which there has been so much controversy. In the absence of a distinct statement as to the intervals at which the areas shall be taken I see no reason for preferring either of the above solutions to the other.

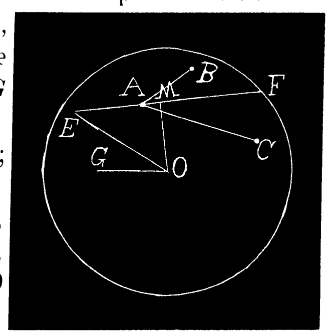
The editor quotes me as saying that there is no correct solution of the problem. I must have failed in expressing myself clearly, for my position is that in such problems no solution can be considered a full one that does not discuss all possible cases.

51. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Three points are taken at random in a sphere and a plane passed through them. Find the average volume of the segment cut off from the sphere.

Solution by the PROPOSER.

Let A, B, C be the three random points, EF the diameter of the section of the sphere made by a plane through A, B, C; M the center of this section; O the center of the sphere; OG a line such that AB is parallel to the plane MOG.



Let $OE=r$, $MA=x$, $AB=y$, $AC=z$, $\angle EOM=\theta$, $\angle BAM=\varphi$, $\angle CAM=\psi$, $\angle MOG=\lambda$, the angle the plane MOG makes with some fixed plane through OG $=\rho$.

The element of the sphere at A is $r\sin\theta d\theta.2\pi x dx$; at B, $y^2dyd\varphi d\lambda$; at C, $\sin(\varphi+\psi)\sin\lambda z^2 dz d\psi d\rho$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of x, 0 and $r\sin\theta$, and tripled; of φ, $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$; of ψ, $-\varphi$ and $\frac{1}{2}\pi$, and doubled; of λ, 0 and π; of ρ, 0 and 2π; of y, 0 and $2x\cos\varphi$; of z, 0 and $2x\cos\psi$.

Let $r\sin\theta = x'$, $2x\cos\varphi = y'$, $2x\cos\psi = z'$, V = volume of segment.

$OM = r\cos\theta$. $\therefore$ The height of the segment is $r(1-\cos\theta) = 2r\sin^2\frac{1}{2}\theta$.

$\therefore$ $V = \frac{4}{3}\pi r^3 \sin^4\frac{1}{2}\theta(3-2\sin^2\frac{1}{2}\theta)$.

Since the whole number of ways the three points can be taken is $(\frac{4}{3}\pi r^3)^3$, the required average is,

$$\Delta = \frac{6.3^3}{64\pi^3 r^9}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\Phi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{y'}\int_0^{z'} Vr\sin\theta d\theta 2\pi x dx$$
$$\times \sin(\varphi+\psi)d\varphi d\psi \sin\lambda d\lambda d\rho y^2 dy z^2 dz.$$

$$\Delta = \frac{27}{2\pi^2 r^8}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\Phi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{y'} V\sin\theta\sin(\varphi+\psi)\cos^3\psi\sin\lambda$$
$$\times d\theta x^4 dx d\varphi d\psi d\lambda d\rho y^2 dy$$

$$= \frac{36}{\pi^2 r^8}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\Phi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi} V\sin\theta\sin(\varphi+\psi)\cos^3\varphi\cos^3\psi\sin\lambda x^7 d\theta dx$$
$$\times d\varphi d\psi d\lambda d\rho$$

$$= \frac{144}{\pi r^8}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\Phi}^{\frac{1}{2}\pi} V\sin\theta\sin(\varphi+\psi)\cos^3\varphi\cos^3\psi x^7 d\theta dx d\varphi d\psi$$

$$= \frac{18}{\pi r^8}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} V\sin\theta[3(\tfrac{1}{2}\pi+\varphi)\sin\varphi + 2\cos\varphi + \sin^2\varphi\cos\varphi]\times\cos^3\varphi x^7 d\theta dx d\varphi$$

$$= \frac{315}{16\pi r^8}\int_0^{\frac{1}{2}\pi}\int_0^{x'} V\sin\theta x^7 d\theta dx$$

$$= \frac{105 r^3}{32}\int_0^{\frac{1}{2}\pi}\sin^4\tfrac{1}{2}\theta(3-2\sin^2\tfrac{1}{2}\theta)\sin^9\theta d\theta = \tfrac{4}{3}r^3.$$

This is the average volume of the lesser segment.

$\frac{4}{3}r^3(\pi-1)$ = average volume of greater.

EDITORIALS.

The MONTHLY will not appear during the months of July and August, but the August-September number will appear about the first of September.

We are pleased to state that our valued contributor, Dr. G. B. M. Zerr, has been called to the presidency of The Russell College, Lebanon, Va. May success, as we know it will, follow him in his new field of work.

The MONTHLY is now sorely in need of funds to carry it on further. Will those of our subscribers who are in arrears remit the amount of their subscriptions at once, so that no delay may be caused through lack of funds in getting out our next issue?

The degree of Doctor of Philosophy was conferred June 9th, by the University of Pennsylvania, on Prof. Robert J. Aley, the subject of his thesis being, "Some Contributions to the Geometry of the Triangle." We congratulate Dr. Aley on having received this degree as it is not an honorary one, but was obtained by actual work done at the University during the past year.

We are sorry that we were obliged to disappoint our readers in failing to give in the May number of the MONTHLY, the first of a series of articles on Lie's Transformation groups, by Dr. Edgar Odell Lovett. Owing to some unavoidable circumstances, Dr. Lovett was unable to prepare the articles, but he assures us that he will have his first article ready for the August-September number. We shall look forward with a good deal of interest for the appearance of the next number.

It has been proposed that the number of pages of the MONTHLY be increased from 32 to 50, half the number of which shall be devoted to papers and the other half to the solutions of problems, and the price of subscription raised to $5. per year. We shall be pleased to hear from every one of our subscribers in regard to this matter, that in case the proposition meets with the necessary endorsement it may be carried into execution at the beginning of the fifth volume. We are at all times open to advice and suggestions from our readers and no pains will be spared on our part to increase the usefulness of our journal.

The University of Chicago, Summer, 1897. The following mathematical courses will be offered :—By Professor Moore : Abstract groups ; Projective geometry.—By Professor Bolza : Hyperelliptic functions ; Advanced integral calculus.—By Dr. Lovett : The geometry of Lie's transformation-groups.—By Dr. Young : [1]Conferences on mathematical pedagogy ; [1]Determinants ; Culture Calculus ; [2]Plane trigonometry.—By Mr. Slaught : Integral Calculus ; College algebra. The courses are four or five hours weekly for twelve weeks from July 1, 1897 ; the two courses marked 1 are, however, only for the first six weeks, and

the course marked 2 is ten hours weekly for the second six weeks. Those who expect to work in mathematics at the University of Chicago during the coming summer, as well as those who desire further information, are requested to communicate with Professor Moore.

It was our intention to have appear in this issue a group of some of our contributors, but it was impossible for us to make all the necessary arrangements without delaying this number. So we have decided to have our group in the August-September number.

We are indebted to Dr. Artemas Martin for pamphlet copies of his valuable papers on "Formulas for the Sides of Rational Plane Triangles," and "A Method of Finding, without Tables, the Number Corresponding to a given Logarithm." These papers will appear in Vol. II., No. 11 of the *Mathematical Magazine*.

We have received a copy in pamphlet form of "Transcendental Numbers," by Prof. Heinrich Weber. Translated into English by Prof. W. W. Beman. Reprinted from the *Bulletin of the American Mathematical Society*. Thanks are due Professor Beman for giving us this reproduction in English of this very interesting and valuable paper.

Ginn & Co. announce for June a *Higher Arithmetic* by Wooster Woodruff Beman, of the University of Michigan, and David Eugene Smith, of the Michigan State Normal School. Teachers will await with much interest this new work on arithmetic by these well-known authors. The same publishers announce as ready "*An Elementary Arithmetic*," by William W. Speer, being the second book of this new series.

BOOKS AND PERIODICALS.

Differential Equations. By D. A. Murray, Ph. D., of the Department of Mathematics in Cornell University. Price $1.90. 230 pages. New York and London: Longmans, Green & Co. 1897.

This work aims to meet the needs of students of physics and engineering who wish to use the subject as a tool, as well as of those students who have more time to give to the general theory and who wish to proceed to the study of the higher mathematics. For the first class, the theoretical explanations have been given as briefly as is consistent with clearness and in most cases the examples have been worked in full detail. In addition, two chapters have been introduced dealing with geometrical and physical problems. For the second class of students, notes have been inserted in the latter part of the book giving the demonstration of additional theorems and more vigorous proofs of theorems partially proved in the first part of the book. Interesting historical and biographical notes have been given in proper places, and many references are made to sources where fuller explanations and developments than the scope of the work allows may be found. We commend the book as providing an excellent introductory course in Differential Equations. J. M. C.

Analytical Geometry. By F. R. Bailey, A. M., and F. S. Woods, Ph. D., Assistant Professors of Mathematics in Massachusetts Institute of Technology. 371 pages. Boston and London : Ginn & Co. 1897.

This book is intended primarily for students in colleges and technical schools. The treatment of subjects included has been complete and rigorous. There are no important departures in method of treatment, but we notice that more space than is usual has been given to the more general form of the equations of the first and second degrees; that the equations of the conics have been derived from a single definition and then by translation of the origin equations of the second degree, wanting the xy term, are handled; and that only later the general equation of the second degree is fully discussed. In solid geometry the treatment is very satisfactory. The examples are numerous and well chosen. No use is made of determinants or calculus—a feature which many will commend and others criticise. Altogether the book is undoubtedly a good one and it should prove a useful text.

J. M. C.

Higher Algebra. By George Lilley, Ph. D., LL. D., Ex-President South Dakota College. 504 pages. Silver, Burdett & Co., New York, Boston and Chicago. 1894.

The first 400 pages are the same as the author's "Elements." As the book only professes to cover the ground required for admission to colleges and universities, this feature is not so objectionable as it would be in a work intended for college and university use. Under the chapter on "Theory of Limits," there are several features which invite attention, such as the proof of the Theory; the sum of an infinite decreasing Geometrical Series; the invention of a symbol to represent an Infinitesimal, etc. However, to our mind the author's interpretation of the for $a/0$, or 0 *as a divisor*, is objectionable, and the proof that, in general. a $0=0$, defective. The proof as given is,

$$\frac{12}{+2}=6,\ \frac{12}{+1}=12,\ \frac{12}{0}=0,\ \frac{12}{-1}=-12,\ \frac{12}{-2}=-6, \text{ etc.},$$

where the quotient, 0, means that there is *no number of times zero* that the divisor, 0, can be subtracted from 12 and leave *zero*. It would misrepresent the author's position not to add that he invents a new symbol to represent an infinitesimal and shows that $a/$(an infinitesimal)$=\infty$, and he would not confound the 0, arising from dividing a by *infinity*, with the *absolute zero*. nor *perhaps* the absolute zero with the zero, meaning "no number of times," in the quotient $a/0=0$. In interpreting the result, $t=a/0$, in Clairaut's problem of the Couriers, he would say, as there is no number of times zero that subtracted from a leaves *zero*, so there is no number of hours when they have been or will be together, and that the form $a/0$ indicates that *the problem is impossible*. That our readers may catch the spirit and meaning of his article, we have invited Dr. Lilley to give some elaboration to his views in a short article for the MONTHLY to be published in a future number. Although we do not approve some of the positions which the author has taken, still we regard the treatise on the whole as one of decided merit. The book has evidently been made for the class room and for actual use, and bears the marks of having been written by an experienced and practical teacher. We have only space to note further the demonstration for "Undetermined Coefficients," on page 419; "Pascal's Arithmetical Triangle," on page 442, which has published in the MONTHLY for December, 1894; and the many interesting notes on the subject of logarithms in the Appendix. J. M. C.

The following periodicals have been received : Journal de Mathématiques Élémentaires, (1er Juin 1897); American Journal of Mathematics, (April, 1897); The Mathematical Gazette, (February, 1897); L' Intermédiare des Mathématiciens, (Mai, 1897); Miscellaneous Notes and Queries, (May, 1897); The Kansas University Quarterly, (January, 1897); The Monist, (April, 1897); Bulletin of the American Mathematical Society, (May, (1897); The Educational Times, (May, 1897), Science, (No. for June 11, 1897); The Review of Reviews, (June, 1897), The Cosmopolitan, (June, 1897); The Arena, (June, 1897).

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

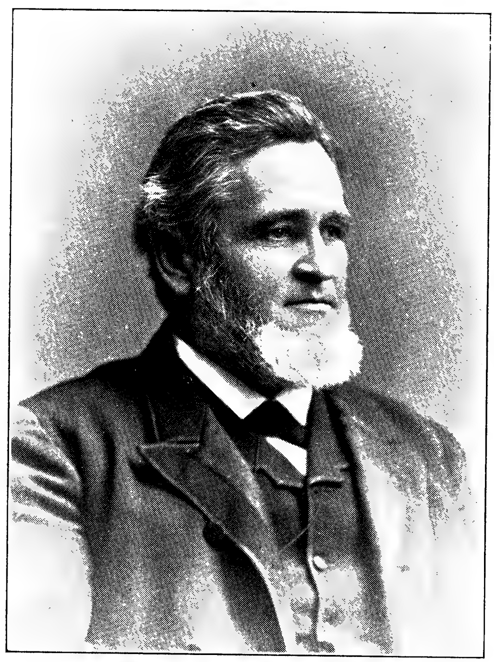
VOL. IV. AUGUST-SEPTEMBER, 1897. Nos. 8-9.

BIOGRAPHY.

PROFESSOR DE VOLSON WOOD.

"PROFESSOR WOOD was a man of wide and enviable reputation. It had been the fortune of many generations of students to sit under his teachings, he had written books which are standard in the technical schools and among engineers, and he had been active all his life in written and spoken discussions before the several societies of which he was a member. Furthermore, he had personal qualities which impressed themselves promptly and strongly upon those who came in contact with him, and as a consequence of all these conditions he was one of the best-known professors in the United States. But beyond all that lay extraordinary ability as a mathematician and as an analyst, remarkable strength and simplicity of character, and a genius for teaching which made his reputation a good deal more than temporary or local.

Professor Wood was a man of considerable practical mechanical ability, but that ability had never been turned to very important results. His powers as a mathematician, however, have given him a permanent place in the literature of engineering, and no student of the higher mathematics of engineering can remain ignorant of the name of DeVolson Wood. But his real greatness was as a teacher. In one sense perhaps that is a misfortune for a man, because he leaves no monument except in the hearts and the minds of the men who actually came under his personal influence. His fame becomes a tradition, fading away and gradually disappearing. On the other hand, is this not the very best work that a man can do in the world—the work of a really strong and sound teacher?



Yours Truly
De Volson Wood.

It would be difficult to sum up in a few words all the qualities which made Professor Wood great as a teacher, but the fundamental quality was his own downright sincerity and his faith in his own work; his mind knew only one test, and that was the truth. To him things were either right or they were wrong, and facts were facts or they were not facts, and he saw no occasion for trying to find any middle ground. But the pursuit of the truth is often enough an arid enterprise, and a man needs more than his own sincerity to get young men to follow him eagerly in that enterprise; and Professor Wood did get his students to work with alacrity, with eagerness, with enthusiasm. A strong element in this was his own rugged and wholesome enthusiasm; another was his air. His solid and robust figure, his keen eye and square jaw, his frank and ready smile—all these were part of his influence on the young men. Added to the genuineness which appeared in all his speech and all his manner was a gift of geniality. The youth who came in contact with him could not help feeling that he stood before a real man, a man strong and sound, mentally and physically; and while youth is not very analytical it is impressed by a man of such quality without knowing why it is impressed. The writer of these words, who had the fortune to sit under Professor Wood four years in civil engineering, can testify that no other teacher ever gave him such hard lessons or ever got out of him so good recitations, and yet there was no sense of hardship in it. It seemed a natural and inevitable thing to work about five times as hard for Professor Wood as for any other teacher, and this perhaps was largely a result of his own enthusiasm in the work. He had furthermore a gift of personal interest in his students. Probably a very small percentage of his pupils—and they must have been unworthy students at that—failed to feel that Professor Wood had a particular personal interest in them. It was not that he took any special trouble with any one man, but he was always able to carry a man's personality in his mind, and he seemed always to be interested in knowing something about a man's career, And so it came about that his influence on the lives of his students did not cease when they left his class-room.

Professor Wood was an active and sincere Christian gentleman, always interested in good work and always exerting a good influence in the community about him. Among a select body of students his name will be known and honored for generations to come as the name of a clear and able writer on the mathematics and mechanics of engineering; among a great body of teachers, students, engineers, and administrators he is remembered in gratitude and love as a strong and wholesome and stimulating friend." *From the Railroad Gazette of July* 2, 1897.

Professor Wood was born near Smyrna, New York, on June 1, 1832, and died at Hoboken, New Jersey, June 27, 1897. He began teaching in 1849, teaching for three terms in Smyrna. In 1853, he graduated from the Albany State Normal School. During the same and the following year he was principal at Napanoch. He was assistant professor of mathematics in Albany Normal, 1854-5, assistant instructor at the Renselaer Polytechnic Institute, Troy, 1855-7,

from which he received the degree of Civil Engineer. Hamilton College conferred the degree of Master of Arts in 1859.

At the University of Michigan he was professor from 1857 to 1872, receiving the degree of Master of Science in the second year of his professorship. Through his labors the department of civil engineering was organized. He became professor of mathematics and mechanics at Stevens Institute of Technology, Hoboken, New Jersey, in 1872, and upon the withdrawal of Prof. R. H. Thurston, to become president of Sibley College, Cornell, he became professor of mechanical engineering, which position he was holding at the time of his death.

He was a member of the American Society of Civil Engineers from 1871 to 1885, also of American Association for the Advancement of Science, since 1879, and its vice president in 1885. He was a member of the American Mathematical Society, and of the Society of Mechanical Engineers, and an honorary member of the Society of Architects. He was the first president of the Society for the Promotion of Engineering Education, started in Chicago at the time of the World's Fair.

He was engineer of the ore-dock, Marquette, Michigan, in 1864, and inventor of a steam rock drill and air compressor.

He contributed articles to the New York Teacher, Johnson's Cyclopædia, Appleton's Cyclopædia of Mechanics, the London Philosophical Magazine, Van-Nostrand's, The American Engineer, Michigan Journal of Education, Journal of Franklin Institute, Railroad Gazette, of which his son is now one of the editors; The Mining and Engineering Journal, Science, The Mathematical Visitor, The Analyst, The Annals of Mathematics, THE AMERICAN MATHEMATICAL MONTHLY, and other magazines.

He was the author of Trusses, Bridges and Roofs, published in 1872, Wood's Edition of Mahan's Civil Engineering, Treatise on the Resistance of Materials, Elements of Analytical Mechanics, Wood's Edition of Magnus' Lessons in Elementary Mechanics, Coördinate Geometry and Quaternions, Key and Supplement to Elements of Mechanics, and to the Mechanics of Fluids, Trigonometry, Turbines, and in 1887 he published one of the greatest of his books, Thermodynamics, which has entered a number of universities and gone through several editions.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from June-July Number.]

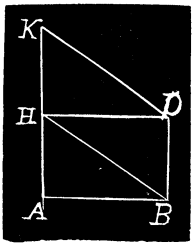
SCHOLION I. And this it is, that I said before in Cor. II. after XXV of this; obviously that no place would remain over for the hypothesis of acute angle, or Euclidean Geometry would be most exactly established, if any two straights existing in the same plane, as suppose AX, BX, which the straight AB meeting (the point B being assumed at a distance from the point A as great as you choose) makes with them toward the same parts of the points X two angles less than two right angles, if (I say) nowhere at another place (this standing) they can admit a common perpendicular.

For then these two AX, BX mutually approach each other ever more, indeed either within a certain determinate limit, as in XXV of this, or without any certain limit, and therefore even to meeting, anyhow after infinite production, as in this XXVII.

But it holds that in either of the aforesaid cases the destruction of the hypothesis of acute angle has now been shown. Quod intendebatur.

SCHOLION II. And again this it is, that I promised at the end of Scholium IV after XXI of this, as from the very terms clearly shines out.

SCHOLION III. Moreover I could wish here to be observed the difference between this proposition and the preceding XVII. For there (recall Fig. 15) has been shown the destruction of the hypothesis of acute angle, if (the straight AB being as small as you choose) every BD erected at whatever acute angle, must at length meet in some point K the perpendicular AH produced.



But here (viceversâ) in fact is permitted the designation of however most small an acute angle at the point A, while still the sect AB to which is to be erected the indefinite perpendicular BX, may be taken of any length whatever.

[To be Continued.]

ON THE COMPLEX ROOTS OF NUMERICAL EQUATIONS OF THE THIRD AND FOURTH DEGREE.

By A. C. BURNHAM, Berlin, Germany.

The real roots of a numerical equation can, as is well known, be found to any desired degree of accuracy by Horner's method of approximation. The complex roots as well can, for cubic and biquadratic equations, be very easily found by the same method. In fact a single application of Horner's method is in these cases sufficient for determining all the roots to any desired number of decimal places, whether the roots be positive or negative, commensurable or incommensurable, real or complex.

THE CUBIC EQUATION.

Let the cubic equation

$$x^3+a_1x^2+a_2x+a_3=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (A),$$

have the roots c, $a+bi$, $a-bi$, since one root must be real, where $i=\sqrt{-1}$ and a, b, c are real. The sums of the products of the roots one, two, and three at a time are equal respectively to $-a_1$, a_2, $-a_3$. That is

$$a+bi+a-bi+c=-a_1,$$

$$(a+bi)(a-bi)+(a+bi)c+(a-bi)c=a_2,$$

$$(a+bi)(a-bi)c=-a_3,$$

or

$$2a+c=-a_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

$$a^2+b^2+2ac=a_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

$$(a^2+b^2)c=-a_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

From these three equations it is not difficult to get the following:

$$8a^3+8a_1a^2+2(a_1^2+a_2)a+a_1a_2-a_3=0 \quad \ldots\ldots\ldots\ldots\ldots \text{I}.$$

$$c=-(a_1+2a) \text{ or } a=-\tfrac{1}{2}(c+a_1) \quad \ldots\ldots\ldots\ldots\ldots \text{II}.$$

$$\pm\sqrt{\frac{a_3-a_1a^2-2a^3}{2a+a_1}}=b=\pm\sqrt{\frac{-a_3}{c}-a^2}=\pm\sqrt{\frac{-a_3-\frac{1}{4}c(c+a_1)^2}{c}} \quad \ldots\ldots \text{III}.$$

Equations I, II, III give all the roots to any desired degree of accuracy. One may find c from the given equation (A) by Horner's method, or a from

equation I by the same method, according to which is the easier. The a or c and b are given by II and III by a mere substitution. It is, of course, immaterial whether the positive or negative value of b be taken, since, in any case, both are used. b will be imaginary only when the original equation (A) has all three roots real. It is also of no consequence which of the three values for a given by equation I be taken, but I will in no case have a greater number of real roots than the given equation (A).

EXAMPLE. Find the roots of $x^3-2x-5=0$.

The one real root c, easily found by Horner's method is, $c=2.0945+$.

We have moreover, $a_1=0$, $a_2=-2$, $a_3=-5$. Therefore, $a=-\frac{1}{2}(c+a_1)$ $=-1.0472+$, and $b=\sqrt{\frac{5}{2.0945}-(-1.0472)^2}=1.123$.

The roots therefore are $-1.0472\pm1.123\sqrt{-1}$ and 2.0945.

In this example, equation I. takes the form

$$8a^3-4a+5=0,$$

which has the one real root $a=1.0472+$. This is the same result as above.

THE BIQUADRATIC EQUATION.

Let the roots of the biquadratic equation

$$x^4+a_1x^3+a_2x^2+a_3x+a_4=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (B).$$

be $a\pm bi$, $c\pm di$. We then have as before

$$a+bi+a-bi+c+di+c-di=-a_1,$$

$$(a+bi)(a-bi)+(a+bi)(c+di)+(a+bi)(c-di)+(a-bi)(c+di)$$
$$+(a-bi)(c-di)+(c+di)(c-di)=a_2,$$

$$(a+bi)(a-bi)(c+di)+(a+bi)(a-bi)(c-di)+(a+bi)(c+di)(c-di)$$
$$+(a-bi)(c+di)(c-di)=-a_3,$$

$$(a+bi)(a-bi)(c+di)(c-di)=a_4,$$

or

$$2(a+c)=-a_1 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

$$(a^2+b^2)+(c^2+d^2)+4ac=a_2 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

$$2c(a^2+b^2)+2a(c^2+d^2)=-a_3 \quad\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

$$(a^2+b^2)(c^2+d^2)=a_4 \quad\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

From these four equations we find

$$a=-\tfrac{1}{2}(2c+a_1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

$$c=-\tfrac{1}{2}(2a+a_1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

$$c^2+d^2=u=\frac{2ca_2+a_3+4c^2(2c+a_1)}{4c+a_1} \ldots\ldots\ldots\ldots\ldots\ldots \text{I}.$$

$$a^2+b^2=t=-\frac{2au+a_3}{2c}=\frac{a_4}{u} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{II}.$$

If we now eliminate u and a by means of 5, I and II, we have the following equation of the sixth degree for c:

$$\begin{aligned}64c^6+96a_1c^5+16(3a_1{}^2+2a_2)c^4+8(4a_1a_2+a_1{}^3)c^3\\ +4(a_2{}^2+2a_1{}^2a_2+a_1a_3-4a_4)c^2+2(a_1a_2{}^2+a_1{}^2a_3-4a_1a_4)c\\ +a_1a_2a_3-a_1{}^2a_4-a_3{}^2=0 \ldots\ldots\ldots\ldots\ldots\ldots \text{III}.\end{aligned}$$

From I and II we have moreover,

$$d=\pm\sqrt{u-c^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

$$b=\pm\sqrt{t-a^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Therefore, after getting a single value of c from III by Horner's method, a, u, t, d, and b follow respectively from (5), I, II, (7), and (8) by mere substitutions, and thus a single application of Horner's method suffices to find all of the roots, no matter what their character.

If the equation (B) has no real roots, then III will have only two real roots. They are separately the values for a and c, and either can be taken for c. That is, the equation of the sixth degree giving a is the same as III giving c.

Example. Find the roots of the equation

$$x^4-6x^3+18x^2-30x+25=0.$$

In this equation $a_1=-6$, $a_2=18$, $a_3=-30$, $a_4=25$.
We have therefore as equation III,

$$4c^6-36c^5+144c^4-324c^3+425c^2-303c+90=0.$$

It is immediately seen that one is a root of this equation, therefore $c=1$, from which there follows,

from (5), $a=-\frac{1}{2}(2-6)=2$,

from I, $u=[2.18-30+4(2-6)]/[-2]=5$,

from II, $t=a_4/u=\frac{25}{5}=5$,

from (7), $d=\pm\sqrt{5-1}=\pm 2$,

from (8), $b=\pm\sqrt{5-4}=\pm 1$.

The roots are therefore $2\pm i$, $1\pm 2i$.

The value 2 for a satisfies the equation III as it should, and 1 and 2 are the only real roots which III possesses.

If the given equation (B) has two real roots and both are known to any desired degree of accuracy, the two other roots are very easily found. Put

$$a+bi=h$$
$$a-bi=k$$

where h and k are known. Then

$$a=\tfrac{1}{2}(h+k),$$
$$b=-\tfrac{1}{2}(h-k)i,$$
$$c=-\tfrac{1}{2}(h+k+a_1) \text{ from (5)},$$

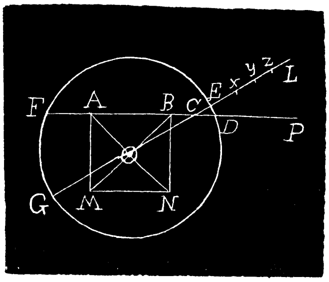
and u is found from I and d from (7) as before. Thus the roots are all determined.

If all of the roots of (B) are real, they will be equally well given by the first method above. In this case b and d will be imaginary.

A DEVICE FOR EXTRACTING THE SQUARE ROOT OF CERTAIN SURD QUANTITIES.

By ROBERT J. ALEY, A. M., Ph. D., Professor of Mathematics, University of Indiana. Bloomington, Indiana.

$ABMN$ is a square. OL is an arm revolving freely about O. This arm beyond C is divided into equal parts at E, x, y, z, etc.



To determine the character of the divisions made on FP by the points of division on OL as OL revolves. Call the side of the square AB, $2a$; BC, b; CE, c; and CD, x.

Then $OC=\sqrt{a^2+(a+b)^2}$.

$$GC=2\sqrt{a^2+(a+b)^2}+c.$$

$FC=2(a+b)+x.$

From the properties of two intersecting chords we have,

$$x\{x+2(a+b)\}=c\{c+2\sqrt{a^2+(a+b)^2}\}$$

$$x^2+2(a+c)x+(a+b)^2=a^2+2ab+b^2+c^2+2c\sqrt{a^2+(a+b)^2}$$

$$x+(a+b)=\sqrt{(a+b)^2+c^2+2c\sqrt{a^2+(a+b)^2}}.$$

Suppose that we examine the results when integral values are given to the constants.

Put $a=c=1$, $b=0$. (Let c take successively the values 1, 2, 3, 4, ... etc.)

Then $x+1=\sqrt{2+2\sqrt{2}}$,

$$x+1=\sqrt{5+4\sqrt{2}},$$

$$x+1=\sqrt{10+6\sqrt{2}},$$

$$x+1=\sqrt{17+8\sqrt{2}}, \text{ etc.},$$

and the law of the series is readily seen.

Put $a=c=b=1$, and let c vary as before.

$$x+2=\sqrt{5+2\sqrt{5}},$$

$$x+2=\sqrt{8+4\sqrt{5}},$$

$$x+2=\sqrt{13+6\sqrt{5}},$$

$$x+2=\sqrt{20+8\sqrt{5}}, \text{ etc.}$$

The law is again evident.

Put $a=1$, $b=2$, and let c vary.

$$x+3=\sqrt{10+2\sqrt{10}},$$

$$x+3=\sqrt{13+4\sqrt{10}},$$

$$x+3=\sqrt{18+6\sqrt{10}},$$

$$x+3=\sqrt{25+8\sqrt{10}}, \text{ etc.}$$

The law is again evident.

Put $a=1$, $b=3$, and let c vary.

$$x+4=\sqrt{17+2\sqrt{17}},$$

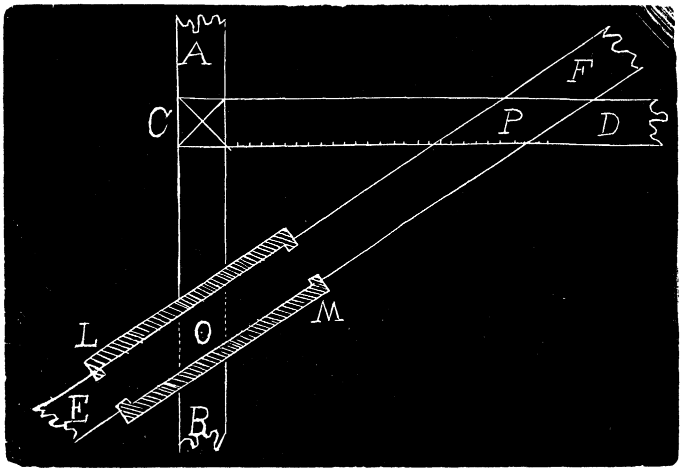
$$x+4=\sqrt{20+4\sqrt{17}},$$

$$x+4=\sqrt{25+6\sqrt{17}}.$$

These examples show how the various series will be found.

From the previous considerations we at once have the data for the construction of a simple mechanical device for the extraction of roots of certain surd quantities.

AB is an upright so arranged that CD will slide up and down always parallel to itself. It is accurately marked to scale so that CD may be set at any de-



sired a. FE works in a slide LM which is free to rotate about O. It is accurately ruled to scale from P to F. By sliding it in LM, P may be set at any desired $a+c$. CD is ruled to scale and is also provided with a diagonal scale, so that by the use of dividers, results may be read to hundredths. When the instrument is set at any chosen a and b, all the roots for that set may be read off at once.

Tables may be easily constructed. A few samples are here given.

The a's are read in the vertical columns, the b's horizontally, and in the squares the c's take successively the values 1, 2, 3, etc. But three terms are given in each square, enough to make the law perfectly evident.

TABLE FOR INTEGRAL VALUES OF a, b, AND c.

$a=$ \ $b=$	0	1	2	3	4	5	6	7	8
1	$\sqrt{2+2\sqrt{2}}$ $\sqrt{5+4\sqrt{2}}$ $\sqrt{10+6\sqrt{2}}$	$\sqrt{5+2\sqrt{5}}$ $\sqrt{8+4\sqrt{5}}$ $\sqrt{13+6\sqrt{5}}$	$\sqrt{10+2\sqrt{10}}$ $\sqrt{13+4\sqrt{10}}$ $\sqrt{18+6\sqrt{10}}$	$\sqrt{17+2\sqrt{17}}$ $\sqrt{20+4\sqrt{17}}$ $\sqrt{25+6\sqrt{17}}$	$\sqrt{26+2\sqrt{26}}$ $\sqrt{29+4\sqrt{26}}$ $\sqrt{34+6\sqrt{26}}$	$\sqrt{37+2\sqrt{37}}$ $\sqrt{40+4\sqrt{37}}$ $\sqrt{45+6\sqrt{37}}$	$\sqrt{50+2\sqrt{50}}$ $\sqrt{53+4\sqrt{50}}$ $\sqrt{58+6\sqrt{50}}$	$\sqrt{65+2\sqrt{65}}$ $\sqrt{68+4\sqrt{65}}$ $\sqrt{73+6\sqrt{65}}$	$\sqrt{82+2\sqrt{82}}$ $\sqrt{85+4\sqrt{82}}$ $\sqrt{90+6\sqrt{82}}$
2	$\sqrt{5+2\sqrt{8}}$ $\sqrt{8+4\sqrt{8}}$ $\sqrt{13+6\sqrt{8}}$	$\sqrt{10+2\sqrt{13}}$ $\sqrt{13+4\sqrt{13}}$ $\sqrt{18+6\sqrt{13}}$	$\sqrt{17+2\sqrt{20}}$ $\sqrt{20+4\sqrt{20}}$ $\sqrt{25+6\sqrt{20}}$	$\sqrt{26+2\sqrt{29}}$ $\sqrt{29+4\sqrt{29}}$ $\sqrt{34+6\sqrt{29}}$	$\sqrt{37+2\sqrt{40}}$ $\sqrt{40+4\sqrt{40}}$ $\sqrt{45+6\sqrt{40}}$	$\sqrt{50+2\sqrt{53}}$ $\sqrt{53+4\sqrt{53}}$ $\sqrt{58+6\sqrt{53}}$	$\sqrt{65+2\sqrt{68}}$ $\sqrt{68+4\sqrt{68}}$ $\sqrt{73+6\sqrt{68}}$	$\sqrt{82+2\sqrt{85}}$ $\sqrt{85+4\sqrt{85}}$ $\sqrt{90+6\sqrt{85}}$	
3	$\sqrt{10+2\sqrt{18}}$ $\sqrt{13+4\sqrt{18}}$ $\sqrt{18+6\sqrt{18}}$	$\sqrt{17+2\sqrt{25}}$ $\sqrt{20+4\sqrt{25}}$ $\sqrt{25+6\sqrt{25}}$	$\sqrt{26+2\sqrt{34}}$ $\sqrt{29+4\sqrt{34}}$ $\sqrt{34+6\sqrt{34}}$	$\sqrt{37+2\sqrt{45}}$ $\sqrt{40+4\sqrt{45}}$ $\sqrt{45+6\sqrt{45}}$	$\sqrt{50+2\sqrt{58}}$ $\sqrt{53+4\sqrt{58}}$ $\sqrt{58+6\sqrt{58}}$	$\sqrt{65+2\sqrt{73}}$ $\sqrt{68+4\sqrt{73}}$ $\sqrt{73+6\sqrt{73}}$	$\sqrt{82+2\sqrt{90}}$ $\sqrt{85+4\sqrt{90}}$ $\sqrt{90+6\sqrt{90}}$		
4	$\sqrt{17+2\sqrt{32}}$ $\sqrt{20+4\sqrt{32}}$ $\sqrt{25+6\sqrt{32}}$	$\sqrt{26+2\sqrt{41}}$ $\sqrt{29+4\sqrt{41}}$ $\sqrt{34+6\sqrt{41}}$	$\sqrt{37+2\sqrt{52}}$ $\sqrt{40+4\sqrt{52}}$ $\sqrt{45+6\sqrt{52}}$	$\sqrt{50+2\sqrt{65}}$ $\sqrt{53+4\sqrt{65}}$ $\sqrt{58+6\sqrt{65}}$	$\sqrt{65+2\sqrt{80}}$ $\sqrt{68+4\sqrt{80}}$ $\sqrt{73+6\sqrt{80}}$	$\sqrt{82+2\sqrt{97}}$ $\sqrt{85+4\sqrt{97}}$ $\sqrt{90+6\sqrt{97}}$			
5	$\sqrt{26+2\sqrt{50}}$ $\sqrt{29+4\sqrt{50}}$ $\sqrt{34+6\sqrt{50}}$	$\sqrt{37+2\sqrt{61}}$ $\sqrt{40+4\sqrt{61}}$ $\sqrt{45+6\sqrt{61}}$	$\sqrt{50+2\sqrt{74}}$ $\sqrt{53+4\sqrt{74}}$ $\sqrt{58+6\sqrt{74}}$	$\sqrt{65+2\sqrt{89}}$ $\sqrt{68+4\sqrt{89}}$ $\sqrt{73+6\sqrt{89}}$	$\sqrt{82+2\sqrt{106}}$ $\sqrt{85+4\sqrt{106}}$ $\sqrt{90+6\sqrt{106}}$				
6	$\sqrt{37+2\sqrt{72}}$ $\sqrt{40+4\sqrt{72}}$ $\sqrt{45+6\sqrt{72}}$	$\sqrt{50+2\sqrt{85}}$ $\sqrt{53+4\sqrt{85}}$ $\sqrt{58+6\sqrt{85}}$	$\sqrt{65+2\sqrt{100}}$ $\sqrt{68+4\sqrt{100}}$ $\sqrt{73+6\sqrt{100}}$	$\sqrt{82+2\sqrt{117}}$ $\sqrt{85+4\sqrt{117}}$ $\sqrt{90+6\sqrt{117}}$					
7	$\sqrt{50+2\sqrt{98}}$ $\sqrt{53+4\sqrt{98}}$ $\sqrt{58+6\sqrt{98}}$	$\sqrt{65+2\sqrt{113}}$ $\sqrt{68+4\sqrt{113}}$ $\sqrt{73+6\sqrt{113}}$	$\sqrt{82+2\sqrt{130}}$ $\sqrt{85+4\sqrt{130}}$ $\sqrt{90+6\sqrt{130}}$						
8	$\sqrt{65+2\sqrt{128}}$ $\sqrt{68+4\sqrt{128}}$ $\sqrt{73+6\sqrt{128}}$	$\sqrt{82+2\sqrt{145}}$ $\sqrt{85+4\sqrt{145}}$ $\sqrt{90+6\sqrt{145}}$							
9	$\sqrt{82+2\sqrt{162}}$ $\sqrt{85+4\sqrt{162}}$ $\sqrt{90+6\sqrt{162}}$								

Fractional Values of a and b.

$b=$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$
$a=\frac{1}{2}$	$\sqrt{2+\sqrt{5}}$ $\sqrt{5+2\sqrt{5}}$ $\sqrt{10+3\sqrt{5}}$	$\sqrt{5+\sqrt{17}}$ $\sqrt{8+2\sqrt{17}}$ $\sqrt{13+3\sqrt{17}}$	$\sqrt{10+\sqrt{37}}$ $\sqrt{13+2\sqrt{37}}$ $\sqrt{18+3\sqrt{37}}$	$\sqrt{17+\sqrt{65}}$ $\sqrt{20+2\sqrt{65}}$ $\sqrt{25+3\sqrt{65}}$	$\sqrt{26+\sqrt{101}}$ $\sqrt{29+2\sqrt{101}}$ $\sqrt{34+3\sqrt{101}}$
$\frac{3}{2}$	$\sqrt{5+\sqrt{25}}$ $\sqrt{8+2\sqrt{25}}$ $\sqrt{13+3\sqrt{25}}$	$\sqrt{10+\sqrt{45}}$ $\sqrt{13+2\sqrt{45}}$ $\sqrt{18+3\sqrt{45}}$	$\sqrt{17+\sqrt{73}}$ $\sqrt{20+2\sqrt{73}}$ $\sqrt{25+3\sqrt{73}}$	$\sqrt{26+\sqrt{109}}$ $\sqrt{29+2\sqrt{109}}$ $\sqrt{34+3\sqrt{109}}$	
$\frac{5}{2}$	$\sqrt{10+\sqrt{61}}$ $\sqrt{13+2\sqrt{61}}$ $\sqrt{18+3\sqrt{61}}$	$\sqrt{17+\sqrt{89}}$ $\sqrt{20+2\sqrt{89}}$ $\sqrt{25+3\sqrt{89}}$	$\sqrt{26+\sqrt{125}}$ $\sqrt{29+2\sqrt{125}}$ $\sqrt{34+3\sqrt{125}}$		
$\frac{7}{2}$	$\sqrt{17+\sqrt{113}}$ $\sqrt{20+2\sqrt{113}}$ $\sqrt{25+3\sqrt{113}}$	$\sqrt{26+\sqrt{149}}$ $\sqrt{29+2\sqrt{149}}$ $\sqrt{34+3\sqrt{149}}$			
$\frac{9}{2}$	$\sqrt{26+\sqrt{181}}$ $\sqrt{29+2\sqrt{181}}$ $\sqrt{34+3\sqrt{181}}$				

Simple Surd Values of a and b.

$b=$	0	$\sqrt{5}$	$2\sqrt{5}$	$3\sqrt{5}$
$a=\sqrt{5}$	$\sqrt{6+2\sqrt{10}}$ $\sqrt{9+4\sqrt{10}}$ $\sqrt{14+6\sqrt{10}}$	$\sqrt{21+2\sqrt{25}}$ $\sqrt{24+4\sqrt{25}}$ $\sqrt{29+6\sqrt{25}}$	$\sqrt{46+2\sqrt{50}}$ $\sqrt{49+4\sqrt{50}}$ $\sqrt{54+6\sqrt{50}}$	$\sqrt{81+2\sqrt{85}}$ $\sqrt{84+4\sqrt{85}}$ $\sqrt{89+6\sqrt{85}}$
$2\sqrt{5}$	$\sqrt{21+2\sqrt{40}}$ $\sqrt{24+4\sqrt{40}}$ $\sqrt{29+6\sqrt{40}}$	$\sqrt{46+2\sqrt{65}}$ $\sqrt{49+4\sqrt{65}}$ $\sqrt{54+6\sqrt{65}}$	$\sqrt{81+2\sqrt{100}}$ $\sqrt{84+4\sqrt{100}}$ $\sqrt{89+6\sqrt{100}}$	
$3\sqrt{5}$	$\sqrt{46+2\sqrt{90}}$ $\sqrt{49+4\sqrt{90}}$ $\sqrt{54+6\sqrt{90}}$	$\sqrt{81+2\sqrt{125}}$ $\sqrt{84+4\sqrt{125}}$ $\sqrt{89+6\sqrt{125}}$		
$4\sqrt{5}$	$\sqrt{81+2\sqrt{160}}$ $\sqrt{84+4\sqrt{160}}$ $\sqrt{89+6\sqrt{160}}$			

Simple Surd Values of a and b.

$b=$	0	$\sqrt{2}$	$2\sqrt{2}$	$3\sqrt{2}$
$a=\sqrt{2}$	$\sqrt{3+2\sqrt{4}}$ $\sqrt{6+4\sqrt{4}}$ $\sqrt{11+6\sqrt{4}}$	$\sqrt{9+2\sqrt{10}}$ $\sqrt{12+4\sqrt{10}}$ $\sqrt{17+6\sqrt{10}}$	$\sqrt{19+2\sqrt{20}}$ $\sqrt{22+4\sqrt{20}}$ $\sqrt{25+6\sqrt{20}}$	$\sqrt{33+2\sqrt{34}}$ $\sqrt{36+4\sqrt{34}}$ $\sqrt{41+6\sqrt{34}}$
$2\sqrt{2}$	$\sqrt{9+2\sqrt{16}}$ $\sqrt{12+4\sqrt{16}}$ $\sqrt{17+6\sqrt{16}}$	$\sqrt{19+2\sqrt{26}}$ $\sqrt{22+4\sqrt{26}}$ $\sqrt{25+6\sqrt{26}}$	$\sqrt{33+2\sqrt{40}}$ $\sqrt{36+4\sqrt{40}}$ $\sqrt{41+6\sqrt{40}}$	
$3\sqrt{2}$	$\sqrt{19+2\sqrt{36}}$ $\sqrt{22+4\sqrt{36}}$ $\sqrt{27+6\sqrt{36}}$	$\sqrt{33+2\sqrt{50}}$ $\sqrt{36+4\sqrt{50}}$ $\sqrt{41+6\sqrt{50}}$		
$4\sqrt{2}$	$\sqrt{33+2\sqrt{64}}$ $\sqrt{36+4\sqrt{64}}$ $\sqrt{41+6\sqrt{64}}$			

Simple Surd Values of a and b.

$b=$	0	$\sqrt{3}$	$2\sqrt{3}$	$3\sqrt{3}$
$a=\sqrt{3}$	$\sqrt{4+2\sqrt{6}}$ $\sqrt{7+4\sqrt{6}}$ $\sqrt{12+6\sqrt{6}}$	$\sqrt{13+2\sqrt{15}}$ $\sqrt{16+4\sqrt{15}}$ $\sqrt{21+6\sqrt{15}}$	$\sqrt{28+2\sqrt{30}}$ $\sqrt{31+4\sqrt{30}}$ $\sqrt{36+6\sqrt{30}}$	$\sqrt{49+2\sqrt{51}}$ $\sqrt{52+4\sqrt{51}}$ $\sqrt{57+6\sqrt{51}}$
$2\sqrt{3}$	$\sqrt{13+2\sqrt{24}}$ $\sqrt{16+4\sqrt{24}}$ $\sqrt{21+6\sqrt{24}}$	$\sqrt{28+2\sqrt{39}}$ $\sqrt{31+4\sqrt{39}}$ $\sqrt{36+6\sqrt{39}}$	$\sqrt{49+2\sqrt{60}}$ $\sqrt{52+4\sqrt{60}}$ $\sqrt{57+6\sqrt{60}}$	
$3\sqrt{3}$	$\sqrt{28+2\sqrt{54}}$ $\sqrt{31+4\sqrt{54}}$ $\sqrt{36+6\sqrt{54}}$	$\sqrt{49+2\sqrt{75}}$ $\sqrt{52+4\sqrt{75}}$ $\sqrt{57+6\sqrt{75}}$		
$4\sqrt{3}$	$\sqrt{49+2\sqrt{96}}$ $\sqrt{51+4\sqrt{96}}$ $\sqrt{56+6\sqrt{96}}$			

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

81. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

How far will a body fall in the first second on the sun, the density of the sun being 25 times that of the earth and its diameter 866400 miles?

Solution by G. B. M. ZERR, A. M., Ph. D., President of Russell College, Lebanon, Va.; and E. W. MORRELL, A. M., Professor of Mathematics, Montpelier Seminary, Montpelier, Vt.

Let G=gravity on sun, g=gravity on earth.
D=density of sun, d=density of earth.
R=radius of sun, r=radius of earth.
Then $G:g=DR:dr$. $\therefore G=gDR/dr$.
Now $g=32.2$, $D=.25d=\frac{1}{4}d$, $R=109.5r$.

$$\therefore G=\frac{32.2\times109.5}{4}=881.475.$$

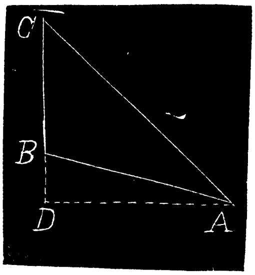
$\frac{1}{2}G=440.7375$ feet, the distance a body will fall the first second.

82. Proposed by CHAS. C. CROSS. Laytonsville, Md.

Two men, A and B, started from the same point at the same time; A traveled southeast for 10 hours and at the rate of 10 miles per hour, and B due south for the same time, going 6 miles per hour; they then turned and traveled directly towards each other at the same rates respectively, till they met. How far did each man travel?

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; G. B. M. ZERR, A. M., Ph. D., President of Russell College, Lebanon, Va.; P. S. BERG, Larimore. N. D.; and H. C. WILKES, Skull Run, Va.

Let C be the starting point; CA=the distance A traveled southeast, and CB=the distance B traveled south. Then $CA=100$ miles, and $CB=60$ miles. Now draw AD perpendicular to CB produced to D. As $\angle D$=a right angle, and $\angle C=45^\circ$, then $CD=AD$.



Then $2\overline{AD}^2=AC^2=100^2$; whence $AD=50\sqrt{2}$, and $BD=50\sqrt{2}-60$. $\therefore$ From the right triangle ADB,

$$AB=\sqrt{(50\sqrt{2})^2+(50\sqrt{2}-60)^2}=\sqrt{13600-6000\sqrt{2}}$$

$$=71.517261+\text{ miles.}$$

A and B together travel 16 miles per hour, and the time required, until they meet in traveling AB, is $\frac{1}{16}$ of $AB=4.469828+$ hours. Therefore, A traveled 44.69828+ miles and B, 26.81897+ miles of the distance AB.

$\therefore$ The total distance traveled by A is 144.69828+ miles, and by B, 86.81897+ miles.

This problem was also solved by *F. R. HONEY*, *C. A. JONES*, and *E. W. MORRELL*.

Solutions of problem 80 were received from G. B. M. Zerr, P. S. Berg, E. W. Morrell, F. R. Honey, and H. C. Wilkes.

NOTE. Hon. Josiah H. Drummond says, in reference to problem 78: "How can you make $986\times3+569=3905$? The question is erroneously enunciated or erroneously solved, or both."

If we assume that the problem is correctly stated, then certainly the published solution is not the solution of the problem. The following is an algebraic statement of the problem as proposed:

Let x=number of cows. Then $3x+569$=number of horses. Let y=number of sheep. Then $4y-126$= number of cows. Hence, $x=4y-126$, $3x=12y-378$, and $3x+569=12y-378+569=12y+191$, the number of horses expressed in terms of the number of sheep. Hence, y, the number of sheep, $+4y-126$, the number cows, $+12y+191$, the number of horses, or $17y+65$=total number=5169. Solving this equation, we do not obtain integral results. If 9 were changed to 5, then y would be integral, and the problem possible. We failed to find this problem in Brooks' Higher Arithmetic. EDITOR.

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

72. Proposed by CHAS. C. CROSS, Laytonsville, Md.

Prove that $\dfrac{2\sqrt{2+\sqrt{3}}}{4+\sqrt{6}-\sqrt{2}}=\sqrt{6}-\sqrt{2}+\sqrt{3}-2$, when reduced to its lowest terms.

I. Solution by JOSIAH H. DRUMMOND, Portland, Maine.

$$\frac{2\sqrt{2+\sqrt{3}}}{4+\sqrt{6}-\sqrt{2}}=\frac{\sqrt{2}\sqrt{4+2\sqrt{3}}}{4+\sqrt{2}(\sqrt{3}-1)},\ =\frac{1+\sqrt{3}}{2\sqrt{2}+\sqrt{3}-1},$$

$$=\frac{(1+\sqrt{3})(2\sqrt{2}-\sqrt{3}+1)}{2(2+\sqrt{3})},\ =\frac{(2-\sqrt{3})(1+\sqrt{3})(2\sqrt{2}-\sqrt{3}+1)}{2},$$

$$=\frac{(\sqrt{3}-1)(2\sqrt{2}-\sqrt{3}+1)}{2},\ =\sqrt{6}-\sqrt{2}+\sqrt{3}-2.$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Russell College, Lebanon, Va.; and P. S. BERG, Principal of Schools, Larimore, N. D.

$$\frac{2\sqrt{2+\sqrt{3}}}{4+\sqrt{6}-\sqrt{2}}=\frac{\sqrt{6}+\sqrt{2}}{4+\sqrt{6}-\sqrt{2}},\ =\frac{(\sqrt{6}+\sqrt{2})(4+\sqrt{6}+\sqrt{2})}{(4+\sqrt{6})^2-2},$$

$$=\frac{\sqrt{6}+\sqrt{3}+\sqrt{2}+2}{5+2\sqrt{6}}, =\frac{(\sqrt{6}+\sqrt{3}+\sqrt{2}+2)(5-2\sqrt{6})}{25-24}$$

$$=\sqrt{6}-\sqrt{2}+\sqrt{3}-2.$$

Also solved by *COOPER D. SCHMITT* and the *PROPOSER.*

73. Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, Russell College, Lebanon, Va.

Find the worth of each of five persons, A, B, C, D, and E, knowing, 1st, that when A's worth is added to a times what B, C, D, and E are worth, it is equal to m; 2nd, when B's worth is added to b times what A, C, D, and E are worth, it is equal to n; 3rd, when C's worth is added to c times what A, B, D, and E are worth, it is equal to p; 4th, when D's worth is added to d times what A, B, C, and E are worth, it is equal to q; 5th, when E's worth is added to e times what A, B, C, and D are worth, it is equal to r.

I. Solution by the PROPOSER.

Let x, y, z, u, v be the worth of A, B, C, D, and E, respectively. Then

$$\begin{aligned} x+a(y+z+u+v)&=m,\\ y+b(x+z+u+v)&=n,\\ z+c(x+y+u+v)&=p,\\ u+d(x+y+z+v)&=q,\\ v+e(x+y+z+u)&=r. \end{aligned}$$

Let $x+y+z+u+v=s$; then $x+a(s-x)=m$.

$\therefore\ x=(m-as)/(1-a)$..........(1). Similarly, $y=(n-bs)/(1-b)$........(2),

$z=(p-cs)/(1-c)$................(3), $u=(q-ds)/(1-d)$................(4),

$v=(r-es)/(1-e)$................(5).

Adding (1), (2), (3), (4), and (5), we get

$$s=\frac{m-as}{1-a}+\frac{n-bs}{1-b}+\frac{p-cs}{1-c}+\frac{q-ds}{1-d}+\frac{r-es}{1-e}.$$

$$\therefore\ s=\frac{\left\{\dfrac{m}{1-a}+\dfrac{n}{1-b}+\dfrac{p}{1-c}+\dfrac{q}{1-d}+\dfrac{r}{1-e}\right\}}{\left\{1+\dfrac{a}{1-a}+\dfrac{b}{1-b}+\dfrac{c}{1-c}+\dfrac{d}{1-d}+\dfrac{e}{1-e}\right\}}.$$

This value of s in (1), (2), (3), (4), (5) gives x, y, z, u, v.

II. **Solution by COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Knoxvflle, Tenn.; and Professor CHAS. C. CROSS, Laytonsville, Md.**

By the conditions we have at once the five equations ; letting x, y, z, u, t, $=A$, B, C, D, E, and F's shares, respectively :

$$\begin{aligned}
&x+ay+az+au+at=m,\\
&bx+y+by+bu+bt=n,\\
&cx+cy+z+cu+ct=p,\\
&dx+dy+dz+u+dt=q,\\
&ex+ev+ez+ev+t=r.
\end{aligned}$$

Hence by Determinants,

$$x=\begin{vmatrix} m & a & a & a & a\\ n & 1 & b & b & b\\ p & c & 1 & c & c\\ q & d & d & 1 & d\\ r & e & e & e & 1\end{vmatrix} \div \begin{vmatrix} 1 & a & a & a & a\\ b & 1 & b & b & b\\ c & c & 1 & c & c\\ d & d & d & 1 & d\\ e & e & e & e & 1\end{vmatrix},$$

$$y=\begin{vmatrix} 1 & m & a & a & a\\ b & n & b & b & b\\ c & p & 1 & c & c\\ d & q & d & 1 & d\\ e & r & e & e & 1\end{vmatrix} \div \begin{vmatrix} 1 & a & a & a & a\\ b & 1 & b & b & b\\ c & c & 1 & c & c\\ d & d & d & 1 & d\\ e & e & e & e & 1\end{vmatrix},$$

and so with z, u, and t, each determinant possessing 120 terms, when expanded.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

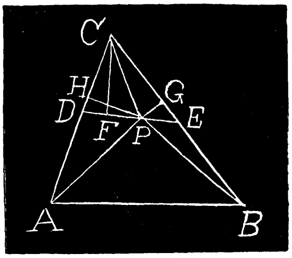
SOLUTIONS OF PROBLEMS.

76. **Proposed by L. B. FRAKER. Bowling Green, Ohio.**

Lines run from a point, P, within a triangular piece of land to the angles A, B, and C are 91, 102, and 80 rods, respectively; and a line 78 rods in length passing through the point, P, and terminating in the sides AC and BC cuts off 3024 square rods adjacent to angle C. Required the dimensions of the land.

Solution by CHAS. C. CROSS, Laytonsville, Md.

Let ABC be the required triangle, $AP=a=91$ rods, $BP=b=102$ rods, $CP=c=80$ rods, $DE=d=78$ rods, the line drawn through P and cutting off 3024 square rods adjacent to the angle C, $CD=x$, $CE=y$, $PE=z$, and area of triangle $DEC=k$.



Draw the perpendiculars PG and PH, from the point P to the sides BC and AC respectively, and draw CF perpendicular to DE. Then

$$x^2=y^2+d^2\pm 2d\times FE, \text{ whence}$$

$$FE=\frac{y^2+d^2-x^2}{2d}. \quad \text{Also } FE=\pm\sqrt{y^2-\frac{4k^2}{d^2}}.$$

$$\text{Hence, } \mp\frac{y^2+d^2-x^2}{2d}=\pm\sqrt{y^2-\frac{4k^2}{d^2}},$$

whence $-(x^2-y^2)^2+2d^2(x^2+y^2)=16k^2+d^2$, an equation containing two unknown quantities. Hence since no other conditions are given by which x or y can be found, it follows that the problem is indeterminate.

By trial, we find that $x=90$ and $y=84$ satisfies the above equation.

Hence, these values furnish a solution, in positive integers, of the problem. Then

$$PF=\sqrt{c^2-\frac{4k^2}{d^2}}=19\tfrac{9}{13} \text{ rods, and } FE=\sqrt{y^2-\frac{4k^2}{d^2}}=32\tfrac{4}{13} \text{ rods.}$$

Hence, $z=PF+FE=52$ rods.

$$BG=\frac{b^2+BC^2-c^2}{2BC} \text{ and } CG=\frac{c^2+y^2-z^2}{2y}. \quad \text{But } BG+CG=BC. \quad \text{Hence,}$$

$$BC=\frac{c^2+y^2-z^2\pm\sqrt{4(b^2-c^2)y^2+(c^2+y^2-z^2)^2}}{2y}=154 \text{ rods.}$$

By similar reasoning with the triangles APC and DPC, we find that $AC=165$ rods.

$$\cos ACB=(90^2+84^2-18^2)\div 2.90.84=\tfrac{3}{5}.$$

$$AB=\sqrt{AC^2+BC^2-2AC.BC\times\cos ACB}=143 \text{ rods.}$$

Hence, the dimensions of the field are $AB=143$ rods, $BC=154$ rods, and $AC=165$ rods.

This problem was proved indeterminate by A. H. Bell.

77. Proposed by CHAS. C. CROSS, Laytonsville, Md.

A line is drawn perpendicular to BC, of the triangle ABC, whose sides are $BC=a$, $CA=b$, and $AB=c$, through A to D, a distance d, (d being equal to or greater than $a+b$); from D a line is drawn to E, a distance e, (e being equal to or greater than $a+b+c$) on BC extended. Required the area of the ellipse which is isogonal conjugate to the straight line DE with respect to the triangle ABC.

I. Solution by G. B. M. ZERR, A. M., Ph. D., President of Russell College, Lebanon, Va.

Using trilinear coördinates and letting F be the point where $AD=d$ cuts BC, we get $DF=(d-b\sin C)$, $EF=\sqrt{e^2-(d-b\sin C)^2}=f$.

$\therefore$ The coördinates of D, E are respectively,

$$\{-(d-b\sin C),\ e\cos C,\ e\cos B\} \text{ and}$$
$$\{0,\ -(f-b\cos C)\sin C,\ -(f+c\cos B)\sin B\}.$$

Let $l=e\{(f-b\cos C)\sin C\cos C-(f+c\cos B)\sin B\cos B\}$,

$m=-(d-b\sin C)(f+c\cos B)\sin B$, and

$n=(d-b\sin C)(f-b\cos C)\sin C$.

Then $l\alpha+m\beta+n\gamma=0$, is the equation to DE, and $l\beta\gamma+m\gamma\alpha+n\alpha\beta=0$, is the equation to the ellipse isogonal conjugate to DE.

Let B be the origin, BC, BA the axes of (x, y).

Then $\alpha=y\sin B$, $\gamma=x\sin B$, $\beta=(ac\sin B-a\alpha-c\gamma)/b$.

$\therefore$ $\beta=\sin B(ac-ay-cx)/b$.

Substituting these values of α, β, γ the equation to the ellipse becomes,

$$clx^2+any^2+(al+cn-bm)xy-aclx-acny=0.$$

Let $\Delta=\dfrac{a^2c^2ln(2-cn-ae)}{4acln-(al+cn-bm)^2}$ be the discriminant of the ellipse.

The two values of z in the equation,

$$z^2+\frac{16(cl+an)\Delta}{\{4acln-(al+cn-bm)^2\}^2}z-\frac{64\Delta^2}{\{4acln-(al+cn-bm)^2\}^3}=0,$$

give the values of the squares of the semi-axes.

$\therefore$ Area of ellipse

$$=\frac{8\pi\Delta}{\{4acln-(al+cn-bm)^2\}^{\frac{3}{2}}}=\frac{8\pi a^2c^2ln(2-cn-al)}{\{4acln-(al+cn-bm)^2\}^{\frac{5}{2}}}.$$

II. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, Ohio.

The coördinates of D are $(d,\ -\beta_1,\ -\gamma_1)$, and of E, $(0,\ \beta_2,\ -\gamma_2)$, and we then have,

$$e^2=(abc)/(4\Delta^2)\{a(\beta_1+\beta_2)(\gamma_2-\gamma_1)+bd(\gamma_1-\gamma_2)+cd(\beta_1+\beta_2)\}\ \ldots\ldots\ldots\ldots\ (1).$$

The equation to the perpendicular to BC through A is

$$\gamma\cos C-\alpha\cos A=0\ \ldots\ldots\ldots\ldots\ (2),$$

and this, passing through D, gives

$$\gamma_1\cos C+d\cos A=0\ \ldots\ldots\ldots\ldots\ (3).$$

We have the constant relation

$$a\alpha+b\beta+c\gamma=2\Delta\ \ldots\ldots\ldots\ldots\ (4),$$

and this being satisfied by the coördinates of D and E,

$$ad-b\beta_1-c\gamma_1=2\Delta\ \ldots\ldots\ldots\ldots\ (5).$$

$$b\beta_2+c\gamma_2=2\Delta\ \ldots\ldots\ldots\ldots\ (6).$$

The equation to DE is

$$\alpha(\beta_1\gamma_2+\beta_2\gamma_1)+\beta d\gamma_2+\gamma d\beta_2=0\ \ldots\ldots\ldots\ldots\ (7),$$

the isogonal conjugate of which is

$$\beta\gamma(\beta_1\gamma_2+\beta_2\gamma_1)+\alpha\gamma d\gamma_2+\alpha\beta d\beta_2=0\ \ldots\ldots\ldots\ldots\ (8),$$

which by the problem is an ellipse.

The area of (8) is expressed by

$$K=2\pi\Delta abc\left\{\begin{vmatrix}0, & \tfrac{1}{2}d\beta_2, & \tfrac{1}{2}d\gamma_2\\ \tfrac{1}{2}d\beta_2, & 0, & \tfrac{1}{2}(\beta_1\gamma_2+\beta_2\gamma_1)\\ \tfrac{1}{2}d\gamma_2, & \tfrac{1}{2}(\beta_1\gamma_2+\beta_2\gamma_1), & 0\end{vmatrix}\div\right.$$

$$\left.\begin{vmatrix}0, & \tfrac{1}{2}d\beta_2, & \tfrac{1}{2}d\gamma_2, & -a\\ \tfrac{1}{2}d\beta_2, & 0, & \tfrac{1}{2}(\beta_1\gamma_2+\beta_2\gamma_1), & -b\\ \tfrac{1}{2}d\gamma_2, & \tfrac{1}{2}(\beta_1\gamma_2+\beta_2\gamma_1), & 0. & -c\\ a, & b, & c, & 0\end{vmatrix}^{\frac{3}{2}}\right\}\ \ldots\ (9).$$

β_1, γ_1 are determined by (3) and (5), and then β_2 and γ_2 from (1) and (6), giving K in terms of d and elements of the triangle of reference.

It is not obvious how much of a reduction (9) admits, and I have not attempted any.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

59. Proposed by MOSES COBB STEVENS, M. A., Department of Mathematics, Purdue University, Lafayette, Ind.

Solve $n\dfrac{d^2y}{dx^2}(x^2+y^2)^{\frac{1}{2}}=\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$.

[From *Forsyth's Differential Equations.*]

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio State University, Athens, Ohio.

Let $y/x=z$, $d^2y/dx^2=v/x$; then the given equation becomes

$$nv\sqrt{1+z^2}=\{1+p^2\}^{\frac{3}{2}},$$

$$\text{or } v=\frac{\{1+p^2\}^{\frac{3}{2}}}{n\sqrt{1+z^2}},=(p-z)\frac{dp}{dz}\ldots\ldots\ldots\ldots\ldots\ldots(A).$$

Assuming $p=(z-t)/(1+zt)$, (A) reduces to

$$\frac{t(dt/dz)}{[1+t^2][t+(1/n)\sqrt{1+t^2}]}-\frac{1}{1+z^2}=0\ \ldots\ldots\ldots\ldots\ldots(B),$$

in which the variables are separated.

II. Solution by G. B. M. ZERR, M. A., Ph D., President and Professor of Mathematics, Russell College, Lebanon, Va.

Let $x=r\cos\theta$, $y=r\sin\theta$, then the equation becomes,

$$nr^3+2nr\left(\frac{dr}{d\theta}\right)^2-nr^2\frac{d^2r}{d\theta^2}=\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Let $\dfrac{dr}{d\theta}=p$, so that $\dfrac{d^2r}{d\theta^2}=p\dfrac{dp}{dr}$;

$\therefore$ (1) becomes, $nr^3+2nrp^2-nr^2p\dfrac{dp}{dr}=(r^2+p^2)^{\frac{3}{2}}$.

Let $p=r/v$, so that $\dfrac{dp}{dr}=\dfrac{1}{v}-\dfrac{r}{v^2}\dfrac{dv}{dr}$;

$\therefore$ (2) becomes, $nv(1+v^2)dr+nrdv=(1+v^2)^{\frac{3}{2}}dr$.

$$\therefore\ dr/r=\frac{ndv}{(1+v^2)\{(1+v^2)^{\frac{1}{2}}-nv\}}.$$

$$\therefore\ \log r=\log\left\{\frac{(1+v^2)^{\frac{1}{2}}}{(1+v^2)^{\frac{1}{2}}-nv}\right\}+\log A.$$

$$\therefore\ r=\frac{A(1+v^2)^{\frac{1}{2}}}{(1+v^2)^{\frac{1}{2}}-nv},\ =\frac{A(r^2+p^2)^{\frac{1}{2}}}{(r^2+p^2)^{\frac{1}{2}}-nr}.$$

$$\therefore\ r\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}-nr^2=A\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}.$$

$$\therefore\ d\theta=\pm\frac{(r-A)dr}{r\sqrt{n^2r^2-(r-A)^2}},\ =\pm\frac{(r-A)dr}{nr^2\sqrt{1-\{(r-A)/(nr)\}^2}}.$$

$$\therefore\ \theta=\pm\frac{1}{\sqrt{n^2-1}}\log[(n^2-1)r+A+\sqrt{(n^2-1)\{r^2n^2-(r-A)^2\}}\,]$$
$$\mp\sin^{-1}\left(\frac{r-A}{nr}\right)+B.$$

$$\therefore\ \tan^{-1}(y/x)\pm\sin^{-1}\left(\frac{\sqrt{x^2+y^2}-A}{n\sqrt{x^2+y^2}}\right)-B,=\pm\frac{1}{\sqrt{n^2-1}}\log[(n^2-1)\sqrt{x^2+y^2}$$
$$+A+\sqrt{(n^2-1)\{n^2(x^2+y^2)-(\sqrt{x^2+y^2}-A)^2\}}\,].$$

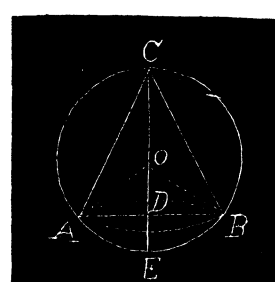
where A and B are constants of integration.

60. Proposed by SETH PRATT, C. E., Assyria, Mich.

To remove $(1/a)$th of the volume of a sphere of a given radius by a conical hole, whose axis is the axis of the sphere, and whose vertex is at the surface of the sphere. Required the height of the cone and the diameter of its base.

I. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let O be the center of the sphere, ABC the cone, $AO=r$, $DE=x$, $AD=y$; then the volume of spherical segment



$$ABE=\frac{\pi x^2}{3}(3r-x),$$

and that of the cone

$$ABC=\frac{\pi y^2(2r-x)}{3};$$

by condition, therefore,

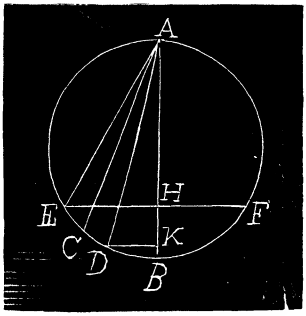
$\dfrac{\pi x^2}{3}(3r-x)+\dfrac{\pi y^2}{3}(2r-x)=(1/a).\frac{4}{3}\pi r^3$. Substituting $y^2=2rx-x^2$, we obtain the final equation, $x^2-4rx=-(4r^2/a)$, whence $x=2r\{1-[1-(1/a)]^{\frac{1}{2}}\}$.

$\therefore$ Height of cone $2r-x=2r[1-(1/a)]^{\frac{1}{2}}$, and $y=2r\{[1-(1/a)]^{\frac{1}{2}}-[1-(1/a)]\}^{\frac{1}{2}}$.

II. **Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Mass.**

Figure shows section of sphere through axis, with $AEBF$ as section of hole.

Let $h=AH=$height of cone ; $b=FH=$radius of base ; $dA=ACD=$elements of area AEB ; $x(=DK)$ is perpendicular to AB ; $\theta=\angle BAD$.



Then $dA=\frac{1}{2}\overline{AD}^2d\theta=2r^2\cos^2\theta d\theta$.

Center of gravity of ADC is at distance $2x/3$ from AB. The element of volume found by revolving ADC about AB, or $dV=2\pi\times(2x/3)\times2r^2\cos^2\theta d\theta$.

But $x=AD\sin\theta=2r\cos\theta\sin\theta$.

$\therefore dV=(16\pi/3)r^3\cos^3\theta\sin\theta d\theta$;

$$\cos\angle EAH=\frac{AH}{AE}=\frac{h}{\sqrt{2rh}}=\sqrt{\frac{h}{2r}}$$

$$\therefore \text{Volume } AEBF=\frac{16\pi r^3}{3}\int_0^{\cos^{-1}\sqrt{(h/2r)}}\cos^3\theta\sin\theta d\theta,=(\pi r/3)(4r^2-h^2),$$

which equals $(1/a)$th of volume of sphere, or $4\pi r^3/3a$.

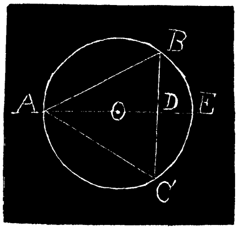
$\therefore h=2r\sqrt{[1-(1/a)]}$.

Diameter$=2b=2\sqrt{h(2r-h)}=4r\sqrt{\sqrt{1-(1/a)}[1-\sqrt{1-(1/a)}]}$.

Volume $AEBF$ can be as easily found by geometry without the use of calculus.

III. **Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, Russell College, Lebanon, Va.**

Let $AO=r$, $DO=y$, $DB=x$. Volume of cone $ABC=\frac{1}{3}\pi(r+y)x^2$; volume of segment $BDCE=\frac{1}{3}\pi(r-y)^2(2r+y)$.



$\therefore \frac{1}{3}\pi(r+y)x^2+\frac{1}{3}\pi(r-y)^2(2r+y)=(4\pi r^3/3a)$, but $x^2=r^2-y^2$.

$\therefore (r+y)(r^2-y^2)+(r-y)^2(2r+y)=(4r^3/a)$.

$\therefore y^2+2ry=(3ar^2-4r^2)/a$.

$$\therefore y=\pm2r\sqrt{\frac{a-1}{a}}-r.$$

The plus sign alone is admissible. $\therefore\ y=2r\sqrt{\dfrac{a-1}{a}}-r.$

$\therefore$ Altitude $=r+y,\ =2r\sqrt{\dfrac{a-1}{a}}$; $2x$=diameter of base$=2\sqrt{r^2-y^2}$.

$$2x=2r\sqrt{\sqrt{\frac{a-1}{a}}-\frac{a-1}{a}},\ =2r\sqrt[4]{\frac{a-1}{a}}\sqrt{\frac{1/\overline{a}-1}{1/a}-\overline{a-1}}.$$

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

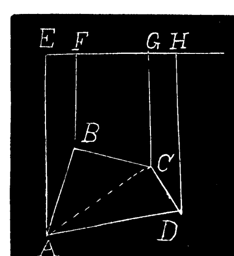
SOLUTIONS OF PROBLEMS.

50. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A plane quadrilateral $ABCD$ in the vertical wall of a cistern, filled with water, has its four vertices A, B, C, D at the distances 10 feet, 4 feet, 5 feet, and 7 feet respectively, from the surface of the water. The projections of AB, BC, CD upon the surface are respectively 2 feet, 3 feet, and 1 foot. Find the pressure of the water upon the quadrilateral, and the position of the center of mean pressure.

Solution by J. C. NAGLE, A. M., M. C. E., Professor of Civil Engineering in the State Agricultural and Mechanical College, College Station, Texas, and the PROPOSER.

In the figure, $AE=10$, $BF=4$, $CG=5$, $DH=7$, $EF=2$, $FG=3$, $GH=1$. The coördinates of the vertices A, B, C, and D with respect to EH and EA as axes are respectively, (10, 0), (2, 4), (5, 5), and (6, 7).



Hence, the area of $ABCD$

$$=\tfrac{1}{2}\left[\begin{vmatrix}x_1 & x_2\\ y_1 & y_2\end{vmatrix}+\begin{vmatrix}x_2 & x_3\\ y_2 & y_3\end{vmatrix}+\begin{vmatrix}x_3 & x_4\\ y_3 & y_4\end{vmatrix}\right]=17\tfrac{1}{2};$$

area of triangle $ABC=10$; and area of $ACD=7\frac{1}{2}$.

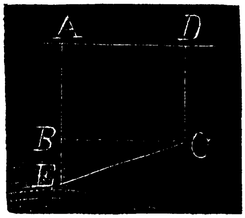
Distance of center of gravity of $\triangle ABC$ from $EH=\frac{1}{3}(10+4+5)=6\frac{1}{3}$, and distance of center of gravity of $\triangle ACD$ from EH $=\frac{1}{3}(10+4+7)=7\frac{1}{3}$. Denoting the distance of center of gravity of $ABCD$ by z, we have $17\frac{1}{2}z=10\times6\frac{1}{3}+7\frac{1}{2}\times7\frac{1}{3}$. Hence, $z=6\frac{16}{21}$. Hence, pressure of water upon $ABCD=17\frac{1}{2}\times6\frac{16}{21}w=\frac{355}{2}w$, w denoting the weight of a cubic foot of water.

For $w=62\frac{1}{2}$ pounds, we find the pressure to be $11093\frac{3}{4}$ pounds.

Let AD represent the surface of the water, $ABCD$ a rectangle, BCE a right triangle, $AE=a$, $CD=b$, $AD=c$. Then, omitting w, the moment of the

pressure upon $ABCD$ with respect to $AD=c\int_0^b x^2dx=\frac{1}{3}b^3c$, and the moment of the pressure upon $BCE=\frac{c}{a-b}\int_0^{a-b}(a-b-x)(x+b)^2dx=\frac{1}{12}c(a-b)(a^2+2ab+3b^2)$.

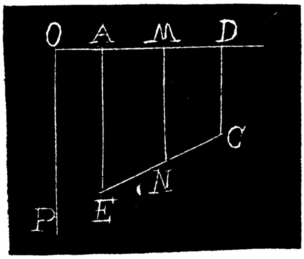
Adding, we find the moment of the pressure upon the trapezoid $AECD$ with respect to $AD=\frac{1}{12}c(a+b)(a^2+b^2)$.



For $ABFE$, $a=10$, $b=4$, $c=2$; $\therefore$ Moment$=270\frac{2}{3}$.
For $FBCG$, $a=4$, $b=5$, $c=3$; $\therefore$ Moment$=92\frac{1}{4}$.
For $GCDH$, $a=5$, $b=7$, $c=1$; $\therefore$ Moment$=74$.
For $EADH$, $a=10$, $b=7$, $c=6$; $\therefore$ Moment$=1266\frac{1}{2}$.

Adding the moments of the first three and then subtracting the sum from the moment of the fourth, we get the moment of $ABCD=829\frac{7}{12}$. Therefore, distance of the center of pressure of $ABCD$ from $EH=829\frac{7}{12}\div\frac{355}{3}=7\frac{3}{284}$.

Let $AECD$ represent a trapezoid with right angles at A and D, AD the surface of the water, OP a perpendicular, and MN a perpendicular to OD ; $AE=a$, $CD=b$, $AD=c$, $OA=h$, $MN=y$, $AM=x$.



$\therefore$ Moment of pressure upon $AECD$ with respect to OP

$$=\frac{1}{2}\int_0^c y^2(x+h)dx,\ \text{where}\ y=\frac{c}{a-b}(a-b-x).$$

Substituting, we get for the moment of $AECD$ with respect to AD the expression $\frac{1}{24}c[c(a^2+2ab+3b^2)+4h(a^2+ab+b^2)]$.

For $ABFE$, $a=10$, $b=4$, $c=2$, $h=0$; $\therefore$ Moment$=38$.
For $BCGF$, $a=4$, $b=5$, $c=3$, $h=2$; $\therefore$ Moment$=110\frac{1}{8}$.
For $CDHG$, $a=5$, $b=7$, $c=1$, $h=5$; $\therefore$ Moment$=100\frac{11}{12}$.
For $ADHE$, $a=10$, $b=7$, $c=6$, $h=0$; $\therefore$ Moment$=580\frac{1}{2}$.

Snbtracting the sum of the first three from the last, we find for the moment of $ABCD$ with respect to AE, $330\frac{11}{24}$.

$\therefore$ Distance of the center of pressure from $AE=330\frac{11}{24}\div\frac{355}{3}=2\frac{2251}{2840}$.

And thus the position of the center of pressure is fully determined.

51. Proposed by **H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.**

"Swift of foot was Hiawatha.
He could shoot an arrow from him
And run forward with such fleetness
That the arrow fell behind him!
Strong of arm was Hiawatha;
He could shoot ten arrows upward
Shoot them with such strength and swiftness
That the tenth had left the bowstring
Ere the first to earth had fallen." *Longfellow*.

Assuming Hiawatha to have been able to shoot an arrow every second and to have aimed when not shooting vertically so that the arrow might have the longest range ; what was Hiawatha's time in a hundred yards ?

I. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Miss.

An arrow rises $4\frac{1}{2}$ seconds when shot vertically, and therefore, the initial velocity which Hiawatha is able to impart to an arrow is $\frac{9}{2}g$ feet per second.

The angle of elevation for the longest range is 45°, and therefore, the horizontal component of the velocity of the arrow is $\frac{1}{4}(9\sqrt{2})g$. This being Hiawatha's speed, his time for 100 yards is a very little less than 3 seconds.

In the above it has been assumed that Hiawatha ran the whole distance at a uniform rate. The range is much more than a hundred yards.

II. Solution by S. ELMER SLOCUM, Union College, Schenectady, N. Y.; J. P. BURDETT, Class '97, Dickinson College, Carlisle, Penn.; and E. W. MORRELL, A. M., Professor of Mathematics, Montpelier Seminary, Montpelier, Vt.

Let t=time of flight when the arrows are shot vertically upward, and u be the initial velocity. Then $t=2u/g$, and $u=\frac{1}{2}gt=144$ feet per second.

The range of a projectile is $u^2\sin 2\theta/g$, and since the greatest value of $\sin 2\theta$ is 1, the maximum range is u^2/g.

$\therefore$ Range$=u^2/g=648$ feet. Time of flight for projectile is $2u\sin\theta/g$ $=6.363$ seconds.

$\therefore$ Velocity$=648\div6.363=101.8+$ feet per second.

Time for 100 yards$=(300\div101.8+)=2.94$ seconds.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

52. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

A straight line of length a is divided into three parts by two points taken at random; find the chance that no part is greater than b. [From *Hall and Knight's Higher Algebra*.]

I. Solution by HENRY HEATON, M. Sc., Atlantic, Iowa.

There are two cases. I, when $b>\frac{1}{3}a$ and $<\frac{1}{2}a$, and II, when $b>\frac{1}{2}a$ and $<a$.

Case I. Let AB represent the line a. Let P be the position of the first point, and let $AP=x$. Lay off PC and BD each$=b$. Then the favorable positions for the second point lie between C and D. $DC=x+2b-a$. The limits of x are $a-2b$ and b.

Hence the required chance is $P_1=\dfrac{1}{a^2}\displaystyle\int_{a-2b}^{b}(x+2b-a)dx=\frac{(3b-a)^2}{2a^2}$.

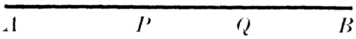
Case II. In this case the limits of x are 0 and $a-b$.

Hence, $P_2=\int_0^{a-b}(x+2b-a)dx=\dfrac{(3b-a)(a-b)}{2a^2}$.

Corollary. When $b=\frac{1}{2}a$, $P_1=P_2=\frac{1}{8}$.

II. **Solution by J. O. MAHONEY, B. E., M. S., Graduate Fellow and Assistant in Mathematics, Vanderbilt University, Nashville, Tenn.**

Let AB be the straight line of length a, and let the random points P, Q be at distances x, y from A, so that $AP=x$, $AQ=y$, and $PQ=a-x-y$. In favorable cases we must have $x<b$, $y<b$, and $a-x-y<b$; and in possible cases $x+y<a$.



Construct the right-angled triangle ABC where $AB=AC=a$. With A as origin and AC and AB as axes construct the lines MN, LH, and RS, whose equations are $y=b$, $x=b$, and $x+y=a-b$, respectively. (1) When $b>\frac{1}{2}a$, the favorable cases will be restricted to the area $MNHLSR$ in Fig. 1, and the required chance is $1-3[(a-b)/a]^2$. (2) When $b<\frac{1}{2}a$, the favorable cases will be restricted to the area 123 in Fig. 2. This is a right-angled isosceles triagle a side of which is $AM-SL=b-(a-2b)=3b-a$. Therefore, the required chance is $[(3b-a)/a]^2$.

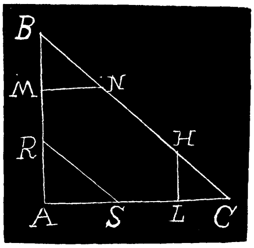


Fig. 1.

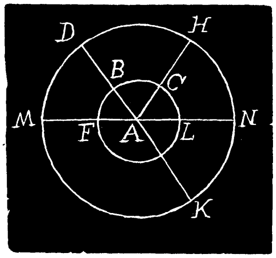
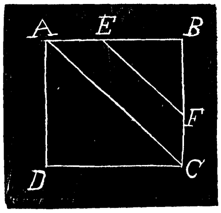


Fig. 2.

III. **Solution by G. B. M. ZERR, M. A., Ph D., President and Professor of Mathematics, Russell College, Lebanon, Va.**

Let $ABCD$ be a square side a, and take $AE=CF=b$. The coördinates of a point taken at random in $ABCD$ are the distances of two such points from one end of the line.



Without restriction the point might fall anywhere upon ABC, but the condition confines it to the triangle EBF.

$$\therefore\ p=\frac{EBF}{ABC}=\frac{(a-b)^2}{a^2}=\left(\frac{a-b}{a}\right)^2.$$

$$\text{Otherwise } p=\frac{\int_b^a\int_0^{y-b}dydx}{\int_0^a\int_0^y dydx}=\left(\frac{a-b}{a}\right)^2.$$

Professors Scheffer and Zerr should have received credit in the last number of the MONTHLY for solving problem 50, and Professor Henry Heaton should have received credit for solving problem 51.

No solution of problem 53 has yet been received.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

45. Proposed by EDWARD R. ROBBINS, Master in Mathematics and Physics, Lawrenceville School, Lawrenceville, New Jersey.

Required several numbers each of which, when divided by 10 leaves a remainder 9; by 9 leaves 8; by 8 leaves 7; by 7 leaves 6; and so on. Also find the least such number which, when divided by 28 leaves 27; by 27 leaves 26; by 26 leaves 25; by 25 leaves 24, *et cetera ad unum.*

I. Solution by M. W. HASKELL, A. M., Ph. D., Department of Mathematics, University of California, Berkley, Cal.; NELSON L. RORAY, Professor of Mathematics in South Jersey Institute, Bridgeton, N. J.; A. H. BELL, Hillsboro, Ill., and H. C. WILKES, Skull Run, W. Va.

The problem can be also stated as follows: Required several numbers each of which, divided by 10, 9, 8, and so on, leaves a remainder (-1).

If then L be the least common multiple of 10, 9, 8 and so on, all numbers of the form $kL-1$, where k is any integer, will have the required character.

Now the least common multiple of 10, 9, 8, ...2, 1 is 2520. The required numbers are then $(2520\,k-1)$ *e. g.*, 2519, 5039, 7559, 10079, etc.

The second problem is solved in exactly the same way. The least common multiple of 28, 27, 26, ...2, 1 is 80313433200. So the required number is one less, or 80313433199.

II. Solution by the PROPOSER.

One less than the product of any number of factors will be divisible by any of the factors, or products of any or all of them, with a remainder one less than the divisor. Because $ab^2c^2d^3-1$ divided by acd gives b^2cd^2-1 for quotient and $acd-1$ for remainder. Thus, the different factors occurring in the natural numbers 1, 2, 3, etc., to 10, are (1.2.2.2.3.3.5.7), one less than the product of which is 2519, which leaves remainders less by unity than the divisors when divided by numbers 1, 2, 3, ...10. All multiples (diminished by one) of the continued product of these factors will satisfy the same demands of the problem, to-wit: 7559, 10079, 12599, etc., etc., *ad libitum.*

The factors occuring in numbers 1......28 are (1.2.2.2.2.3.3.3.5 5.7.11. 13.17.19.23) and one less than their continued product gives 80313433199, the number required.

NOTE. Of course the same numbers will accomodate 5 and 6; 9 and 10; 11 and 12; 13, 14, and 15; 17 and 18; 19, 20, 21, and 22; 23 and 24; 25 and 26; 27 and 28; and so on.

III. Solution by JOSIAH H. DRUMMOND, Portland, Maine.

I. $10a+9$ answers the first condition; multiply this by 9 and add 8, and

we have $90a+89$; proceeding in the same manner we finally have $3,628,800a+3,628,799$, in which a may be zero or any number.

II. Or, in the process as above, we may leave out factors of numbers already used and we reach the result $2520a+2519$, in which a may be zero or any number; if $a=$zero, we have 2519, the smallest number that will answer the conditions of the first question.

III. It is manifest that if we take 1 from a number divisible by all the given divisors, the remainder when divided by those divisors will always leave a remainder one less than the divisor. Hence the least common mnltiple of the given divisors, less 1, is the number required. Hence, omitting the common factors in the second part of the question, we have $28.27.26.25.23.22.19.17....1=80,313,433,199$, the number required.

IV. **Solution by W. H. CARTER, Professor of Mathematics, Centenary College of Louisiana, Jackson, La.**

Let $10x_{10}+9=$the number, also, $9x_9+8$; $8x_8+7$; $7x_7+6$; $6x_6+5$; and so on to $2x_2+1=$the number.

$\therefore 9x_9+8=10x_{10}+9$. $\therefore x_9=x_{10}+\frac{1}{9}(x_{10}+1)$.

But x_9 and x_{10} are both integral.

$\therefore (x_{10}+1)/9=m$ an integer. $\therefore x_{10}=9m-1=(90m/10)-1$.

The value of x_9 from the above equation is $(90m/9)-1$.

Similarly for the other values, the expression $x_n=(90m/n)-1$, giving one of the values for each value of n from 10 to 2. But since all these values are to be integral, $90m$ must be a multiple of each of the natural numbers from 2 to 10 inclusive. This requires m to be $4\times7=28$, or some multiple of 28. If $m=28$, $x_{10}=251$.

$\therefore 10x_{10}+9=2519=$one of the numbers.

Taking $m=$the multiples of 28, we get other numbers, 5039, 7559, 10,079. Still other numbers can be obtained by taking the higher multiples of 28 for m.

A similar solution gives for second statement, the number 80,313,433,199.

V. **Solution by O. W. ANTHONY, M. Sc., Columbian University, 1702 S Street, Washington, D. C.**

The problem in question may be generalized thus: Find a number such that if it be divided by a particular number or any number less than this number the remainder will be one less than the divisor.

Let x be the required number. It is evident, if k and $k+l$ be any two numbers less than the first divisor in question, the following conditions must be satisfied:

$x/k=u_1+(k-1)/k$......(1). $k/(k+l)=u_2+(k+l-1)/(k+l)$ .... (2),

or $x=ku_1+k-1$............(3), and $x=(k+l)u_2+k+l-1$ ....(4).

Take the value of x given in (3) and substitute it for u_2 in (4). Then

$x=(k+l)[ku_1+k-1]+k+l-1\ldots\ldots(5)$, which may be reduced to the following form : $x=k[(k+l)u_1+k+l-1]+k-1\ldots\ldots(6)$.

Thus (5) and (6), which are identical, contain both the forms (3) and (4). Thus if we substitute in the manner indicated the result will contain two original forms. Some special forms required by the problem in question are :

$x=2u_1+1\ldots(1)$; $x=3u_2+2\ldots(2)$; $x=4u_3+3\ldots(3)$; $x=5u_4+4\ldots(4)$;

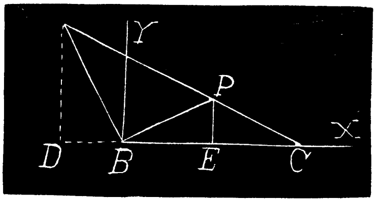
etc., etc. Substitute (1) in (2) in the manner indicated above and we have $x=6u_1+5$. This includes (1) and (2). Substitute this in (3) ; the result is $x=24u_1+23$. This includes (1), (2), and (3) by the previous demonstration. Continuing this we have as a result $x=\underline{|k}u_1+\underline{|k}-1=\underline{|k}(u_1+1)-1\ldots\ldots(A)$. This contains forms (1), (2), (3), (4), etc., and is the general form of number required. The examples cited are special applications of this general form. Thus $x=\underline{|8}(u_1+1)-1$ contains all the numbers required in the first part of the problem, and, letting $u_1=0$, and $k=25$, we have $x=\underline{|25}-1$, the number required in the last part of the problem.

46. Proposed by A. H. HOLMES, Box 963, Brunswick, Maine.

The base BC of the triangle ABC is $2c$, the sum of the two sides, AB and BC, is $2a$. BP is always perpendicular to AB and cuts AC in P. What is the locus of the point P?

I. Solution by GEORGE LILLEY, Ph. D., LL. D., 394 Hall Street, Portland Ore.

Take BC for the axis of x ; let P be (x, y) ; draw AD at right angles to BC, produced ; and PE at right angles to BC.



Area ABP+area PBC=area ABC, or

$(a-c)\sqrt{x^2+y^2}+cy=c\times AD\ldots\ldots\ \ldots\ldots(1)$.

Triangles ABD and BPE are similar.

Hence, $AD=[2y(a-c)]/\sqrt{x^2+y^2}\ldots\ldots(2)$.

From (1) and (2), $(a-c)(x^2+y^2)+cy\sqrt{x^2+y^2}=2cy(a-c)$.

$\therefore\ c^2y^2(x^2+y^2)=(a-c)^2(2cy-x^2-y^2)$, for the required locus.

If $\angle ABC$ be an acute angle, y must be taken negatively. Then, area ABC +area BPC=area ABP, or

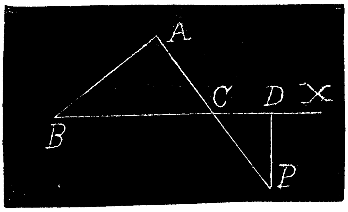
$$c\times AD+c(-y)=(a-c)\sqrt{x^2+y^2}\ \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$\text{and } AD=[2x(a-c)]/\sqrt{x^2+y^2}\ \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

From (3) and (4), $c^2y^2(x^2+y^2)=(a-c)^2(2cx-x^2-y^2)^2$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Lebanon, Va.

Let B be the origin, Bx the initial line, $BP=r$, $\angle CBP=\theta$, $BC=2c$, $AB=2(a-c)$.



Then $\cos B=\sin\theta$.

$AC=\sqrt{8c^2-8ac+4a^2-8c(a-c)\sin\theta}$.

$CP=\sqrt{4c^2-4rc\cos\theta+r^2}$.

$r^2+4(a-c)^2=AP^2=(AC+CP)^2$.

Substituting and reducing we get for the locus

$$4c^2(r\cos\theta+2a\sin\theta-2c-2c\sin\theta)^2$$

$$=(4c^2-4rc\cos\theta+r^2(\{8c^2-8ac+4a^2-8c(a-c)\sin\theta\}.$$

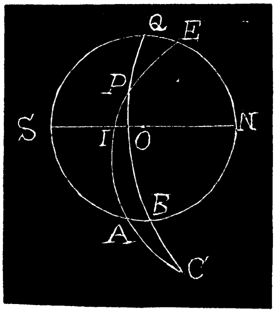
Also solved by *A. H. BELL*.

47. Proposed by S. HART WRIGHT, A. M., Ph. D., Penn Yan, New York.

In longitude 75 degrees west of Greenwich, latitude 43 degrees, 30 minutes north on January 1, 1895, at 3 o'clock A. M., local time. What points of the ecliptic were then rising, setting and on the meridian? Any other necessary data may be taken from an ephemeris.

Solution by the PROPOSER.

January 1, 1895, 3 A. M., in local mean time, at the station, is December 31, 1894, 15th hour astronomical time. And 15h.+5h., the longitude=20h., mean solar time=20h. 3m. 17.1295s. of sidereal time. To this add from ephemeris sidereal time of mean noon at Greenwich, 18h. 39m, 36.83s., and we have 14h. 42m. 52.9595s.$=h$, the sidereal time at station. The vernal equinox is then h hours *west* of the meridian of station, or 24h.$-h$ *east* of it, and therefore $24-(h+6)=$3h. 17m. 7.0405s.$=49^\circ\ 16'\ 45.6''=a$, east of, and below the east point of the horizon of the station.



Let $NQSBN$ be the horizon of station, $EIAC$ a portion of the ecliptic, $QOBC$ a portion of the equator, C the place of the vernal equinox, A the rising point of the ecliptic, I the point then on the meridian, and E the setting point, P the autumnal equinox, and B the point east of the horizon.

Then $BC=a$, the angle BCA, IPO=obliquity of ecliptic$=23^\circ\ 27'\ 19''$ per ephemeris for the date. The angle $ABC=90^\circ+$the latitude of station$=133^\circ\ 30'$. In the spherical triangle ABC, we have, therefore, the angles B and C given and the side $a=49^\circ\ 16'\ 45.6''$ to find the side $AC=b$. By spherical trigonometry, $b=73^\circ\ 45'\ 15''$. In the right spherical triangle IPO, right angled at O, we have $h-12$ hours=2h. 42m. 52.9595s.$=PO=40^\circ\ 43'\ 14.4''$. By spherical trigonometry, $PI=43^\circ\ 10'\ 36''$. Hence the rising point is $360^\circ-b=286^\circ\ 14'\ 45''$ of the

ecliptic, and as great circles intersect in opposite points, E will be 180° less than A, or 106° 14′ 45″, and 180° $+PI$=223° 10′ 36″, the longitude of the point passing the meridian.

The senseless divinations of Astrology, are almost entirely based upon finding the three points of the ecliptic required in this problem, for the moment of birth, at a given place.

Also solved by *EDMUND FISH*, Hillsboro, Ill.

48. Proposed by F. P. MATZ. D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Penn.

In case of *mischance*, with what force would the cow, weighing $w=700$ pounds, jumping over the moon, have struck Her Lunar Majesty in the face?

Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, Russell College, Lebanon, Va.

Let m=mass of cow on moon, $g'=\frac{1}{6}g$=gravity on moon, r=2163 miles=radius of moon, a=238840 miles=distance from earth to moon, A=momentum $=mv$, E=kinetic energy$=\frac{1}{2}mv^2$.

$$\text{Then } v^2=2g'r\left(\frac{a-r}{a}\right),\quad m=\frac{700}{6g'}.$$

$$\therefore\ A=\tfrac{700}{6}\sqrt{\frac{2r}{ag'}(a-r)},=\tfrac{700}{3}\sqrt{\frac{3r}{ag}(a-r)},$$

$$=\tfrac{700}{3}\sqrt{\frac{6489\times5280\times236677}{238840\times32.2}}=239595.79 \text{ foot-pounds.}$$

$E=(350r/3a)(a-r)=1320341350.762$ foot-pounds.

The value of A is the force required.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

83. Proposed by the late REV. G. W. BATES, A. M., Pastor of M. E. Church, Dresden City, Ohio.

A has three notes; the first and second, \$1000 each, and the third \$457; all dated April 1, 1884. The first is due April 1, 1888, second, April 1, 1889, and the third, April 1, 1890, and each bearing interest at 6%. What must B pay for the three notes September 21, 1886 that the investment will bring him 8% compound interest?

[NOTE—The above problem was the result of an actual business transaction.]

84. Proposed by SYLVESTER ROBBINS, North Branch Depot, N. J.

Show how to find sides. integral, fractional, and irrational for twenty-four triangles, each one containing 330 square yards.

85. **Proposed by E. W. MORRELL, A. M., Professor of Mathematics, Montpelier Seminary, Montpelier, Vt.**

In turning a one-horse chaise within a ring of a certain diameter, it was observed that the outer wheel made two turns, while the inner wheel made but one. The wheels were each 4 feet high; and supposing them fixed at the distance of 5 feet on the axletree, what was the circumference of the track described by the outer wheel? From *Greenleaf's National Arithmetic.*

86. **Proposed by EDGAR H. JOHNSON, Professor of Mathematics, Emroy College, Oxford, Ga.**

$$\tfrac{1}{7}=.\dot{1}4285\dot{7}\ ;\ \tfrac{1}{11}=.\dot{0}\dot{9}\ ;\ \tfrac{1}{13}=.\dot{0}7692\dot{3}\ ;\ \tfrac{1}{17}=.\dot{0}58823529411764\dot{7}.$$

Observe that if the numbers forming the first half of the repetend be added respectively to the numbers forming the second half of the repetend, the sum is in every case 9. What is the general law of which these are special cases?

GEOMETRY.

80. **Proposed by J. C. GREGG, Superintendent of Schools, Brazil, Ind.**

One circle touches another internally, and a third circle whose radius is a mean proportional between their radii passes through the point of contact. Prove that the other intersections of the third circle with the first two are in a line parallel to the common tangent of the first two. [From *Phillips and Fisher's Geometry.*]

81. **Proposed by CHAS. C. CROSS, Laytonsville, Md.**

A circle is drawn bisecting the lines joining the points of contact of the inscribed circles with the sides produced. Another circle is drawn passing through the centers of the circles drawn tangent externally to the in-circle and internally to the sides of the triangle. Prove that the centers of these two circles, the incenter and the circumcenter are collinear.

82. **Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Atronomy, Pacific College, Santa Rosa, Cal.**

If the extremities of the base of a triangle be joined by straight lines to the exterior angles of squares constructed upon its two sides, the superior pair of lines thus drawn intersect at right angles; the inferior pair intersect at a point in a line drawn from the vertical angle perpendicular to the base.

MECHANICS.

58. **Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Miss.**

An endless uniform chain is hung over two small smooth pegs in the same horizontal line. Show that, when it is in a position of equilibrium, the ratio of the distance between the vertices of the two catenaries to half the length of the chain is the tangent of half the angle of inclination of the portions near the pegs. [From *Ruth's Analytical Statics. Mathematical Trifos, 1855.*]

59. **Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio State University, Athens, Ohio.**

Find the radius of sphere of given specific gravity which will rest just immersed in a fluid whose density varies as its depth.

60. **Proposed by J. SCHEFFER, A. M., Hagerstown, Md.**

What must be the ratio of the two legs of a uniform and heavy right triangle suspended from the center of the inscribed circle, if this triangle will rest with the shorter leg in a horizontal position?

AVARAGE AND PROBABILITY.

57. Proposed by G. B. M. ZERR, A. M., Ph. D., Presfdent and Professor of Mathematics, Russell College, Lebanon, Va.

A chord is drawd through two points taken at random in the surface of a circle. If a second chord be drawn through two other points taken at random in the surface, find the chance that the quadrilateral formed by joining the extremeties of the two chords will contain the center of the circle.

58. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

From a point on the surface of a circle two lines are drawn to the circumference. Required the average area that may be cut from the circle in this way if the lines are supposed to be drawn at equal angular intervals.

Query I. How does this differ from problem 32?

Query II. Is *sector* the proper word to use for the surface thus cut off?

Query III. It is absolutely correct to use the word *random* in average problems?

NOTES.

THE INTERNATIONAL MATHEMATICAL CONGRESS.

The meeting at Zurich, August 9th–11th, of the International Congress of Mathematicians was in every way a success. More than two hundred members took part. America sent seven representatives, including, however, three Cambridge graduates, now transplanted to Pennsylvania, Professors Harkness, Morley and Charlotte Scott. The greatest mathematician in the world, Sophus Lie, was not expected; and the greatest French mathematician, Poincaré, though down for a speech, did not come; but the actual program was particularly rich and interesting.

It is very noteworthy that the Congress was divided into five sections: (1) Arithmetic and Algebra; (2) Analysis, and Theory of Functions; (3) Geometry; (4) Mechanics and Mathematical Physics; (5) History and Bibliography.

The program of the first section contained the only title in English: "On Pasigraphy, its present state and the pasigraphic movement in Italy," by Ernst Schroeder, of Karlsruhe, author of "Algebra der Logik."

The second section contained a title from Z. de Galdeano, whose heroic efforts gave Spain a Journal of Mathematics, now unfortunately dead in the decadence of that beautiful, priest-ridden land.

The program of the third section, the only one consecrated wholly to a single title, Geometry, contained two titles on the non-Euclidean geometry.

Burali: Les postulats pour la geométrie d'Euclide et de Lobatschewsky.

Andrade: 'La statique non euclidienne et diverses formes mécaniques du postulatum d'Euclide.

In Section IV. Stodola treated an important subject, "Die Beziehungen der Technik zur Mathematik."

In the fifth section Eneström gave an important discussion of bibliography, a point where the Congress can and will render aid of fundamental importance.

In the first general assembly Rudio spoke on the aim and organization of international mathematical congresses.

It was determined that the next Congress should take place at Paris in 1900, under the auspices of the Société Mathématique de France.

As aims were specified : (1) to promote personal relations between mathematicians of different lands ; (2) to give, in reports or conferences, an aperçu of the actual state of the divers branches of mathematics, and to treat questions of recognized importance ; (3) to deliberate on the problems and organization of future congresses ; (4) to treat questions of bibliography, of terminology, etc., on subjects where an *entente international* appears necessary.

Rudio mentioned the yearly issue of an address-book of all the mathematicians of the world with indication of their specialties ; also of a biographic dictionary of living mathematicians with portraits ; also of a literary journal for mathematics.

At the second general assembly Peano gave a conference : "Logica matematica"; and Felix Klein a conference on teaching higher mathematics.

Three important resolutions were introduced by Vasiliev, of Kazan ; Laisant, of Paris, and G. Cantor, of Halle, constituting : (1) a commission for preparation of general reports ; (2) a standing bibliographic and terminology commission ; (3) a commission to give the congress a permanent character by archives, libraries, stations for correspondence, editing or publishing noteworthy works, etc.

Surely this Congress has proven that it came only in the fullness of time, and that the world moves ! GEORGE BRUCE HALSTED.

Austin, Texas.

EDITORIALS.

Dr. O. E. Lovett has been called to Princeton University as Assistant Professor of Mathematics.

Dr. George Lilley, LL. D., has been elected to the Chair of Mathematics in the State University of Oregon.

A portrait of a group of five of our contributors will appear soon. We were unable to complete the arrangements for this number.

Dr. L. E. Dickson, who spent last year at the Universities of Göttingen and Paris, has been elected Assistant Professor of Mathematics in the University of California.

Miss Mary F. Winston, Ph. D., has been elected Professor of Mathematics at the Kansas State Agriculturist College, Manhattan, Kansas.

Prof. E. D. Roe, Jr., Assistant Professor of Mathematics in Oberlin College, is taking a two years course in mathematics, in Göttingen, Germany.

Professor D. A. Lehman, the past year Professor of Mathematics in the College of the Pacific, has been called to the Chair of Mathematics in the Balwin University, Berea, Ohio.

The biography of Professor J. J. Sylvester which appeared in the June-July number of the MONTHLY has been translated in Russian and published by Professor Vasiliev, the great Russian Mathematician.

We regret to record the death of one of our valued contributors, De Volson Wood, Professor of Mechanical Engineering at the Steven Institute of Technology, Hoboken, N. J., on June 27, at the age of sixty-five years. We take pleasure in giving our readers a short account of his life in this issue.

We are pleased to state that we have in our hands Dr. Lovett's first article on Sophus Lie's Transformation Groups, which will surely appear in our next issue. It is Dr. Lovett's purpose to make the series of articles very elementary at first and thus bring this most important subject within the comprehension of the most of our readers. These articles alone will be worth many times the price of subscription to the MONTHLY.

BOOKS AND PERIODICALS.

The Non-Regular Transitive Substitution Groups whose Order is the Product of Three Unequal Prime Numbers. Reprint of a paper in *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich.* By Dr. G. A. Miller, Paris, France. 6 pages. B. F. F.

A History of the United States. By Allen C. Thomas, A. M., Professor of History in Haverford College, Penn. 8vo. cloth and leather back. 418 and lxxiv pages. Boston : D. C. Heath & Co.

This is the best school history of the United States that has yet been published.

B. F. F.

The Tutorial Statics. By William Briggs and G. H. Bryan. 260 pages. Price, $1.00. London : W. B. Clive. New York : Hinds and Noble.

The plan of this work is good and the execution satisfactory. With the exception of some looseness of statement in certain paragraphs, the work is well written and should prove serviceable for class use. There are many valuable hints, explanations and alternative proofs, and a large selection of examples, throughout the text. An excellent summary of results follows each chapter. J. M. C.

Grammar School Arithmetic by Grades. Edited by Eliakim Hastings Moore, Ph. D., Head Professor of Mathematics, The University of Chicago. 8vo. cloth. 352 pages. Price, 60 cents. Chicago: American Book Co.

Some of the prominent features of this work are, the accurate definitions of terms according to modern usage, the use throughout of the inductive or laboratory method, the numerous well selected problems, and the entire absence of rules. The treatment of arithmetic as given in this book is a definite departure from the old ruts, and we believe that the timely appearance of this work will go far towards correcting many of the vicious and unwholesome methods pursued in many schools. B. F. F.

Elementary Text-Book of Physics. By Prof. Wm. A. Anthony, formerly of Cornell University, and Prof. Cyrus F. Brackett, of Princeton University. Revised by Prof. William Frances Magie, of Princeton. Eighth edition, revised. 8vo. cloth. 512 pages. Price, $3.00. New York and London: John Wiley & Sons.

This work deserves especial praise for the direct and logical manner in which it discusses the fundamental principles of Physics. The pictorial representations of apparatus are purposely omitted as are also the illustrations of the fundamental principles by detailed description of special methods of experimentation and of devices necessary for their applications in the arts, and thus space is saved for the discussion of important principles.

The work is admirably adapted to those schools and colleges having a large collection of apparatus, but for those that have but few pieces of apparatus, the absence of pictorial representations in a text book would in many cases leave the student without any ideas at all as to their construction. B. F. F.

Theory of Physics. By Joseph S. Ames, Ph. D., Associate Professor of Physics and Sub-Director of the Physical Laboratory in Johns Hopkins University. Crown 8vo. cloth. 514 pages. Price, $1.60; by mail, $1.75. New York: Harper and Brothers.

"To present successfully the subject of Physics to a class of students, three things seem to me as necessary: a text-book, a course of experimental demonstrations and lectures, accompanied by recitations, and a series of laboratory experiments, mainly quantitative, to be performed by the students themselves under the direction of instructors. I place "text-book" first, because for many reasons I believe it to be the most important of the three. None but advanced students can be trusted to take accurate and sufficient notes of lectures; and a text-book which states the theory of the subject in a clear and logical manner so that recitations can be held on it, seems to me to be absolutely essential." *Preface.*

This work which has just recently been issued discusses in a most satisfactory manner, the latest discoveries made in Physics. The doctrines of energy are stated with the utmost clearness and are made the framework for a consecutive treatment of Physics as a whole. The strong points in favor of this book are too numerous to mention in the limited space at our disposal. B. F. F.

The New Arithmetic. Part Part One for Teachers. By William W. Speer, Assistant Superintendent of Schools, Chicago. 154 pages. Boston and London: Ginn & Co. 1897.

This book is one of a series now in press. Some rather radical departures are proposed. The author thinks that the study of Arithmetic should be advanced from the science of *number* to that of the *definite relations of quantity.* The book gets the idea of *technical measurement* in early. Simple ratios are made the key to the solution of all problems.

The quotations in support of the theory of the book it seems to us are carried to excess. We doubt if the representation of *cents* by *lines*, p. 118, leads to clear ideas of relative val ues, and the "guessing" exercise on page 42 seems rather ludicrous. Notwithstanding minor objections the book is undoubtedly one of many excellencies, and the appearance of the other books of the series will be awaited with more than usual interest. J. M. C.

Mathematical Questions and Solutions. From the "Educational Times," with an Appendix. Edited by W. J. C. Miller, B. A. Vol. LXVI. 128 pages. Francis Hodgson, 89 Farringdon Street, E. C., London.

This valuable reprint contains solutions of 145 interesting problems. The price is 5s. 3d., postpaid. J. M. C.

Descriptive Geometry. Straight Line and Curves. By William J. Meyers, Professor of Mathematics in the State Agricultural College of Colorado, Fort Collins, Colo. Pages, 66 and several pages of excellent Plates. Printed by the Author.

The author has aimed to strike a mean between an abstract and difficult treatment and a diffuse and easy one. The method is based on the authors experience in his class room. The book is well supplied with suitable exercises, and deserves careful examination on the part of teachers who have occasion to use an elementary text on this subject. J. M. C.

Introduction to Infinite Series. By William F. Osgood, Ph. D., Assistant Professor of Mathematics in Harvard University. 71 pages. Cambridge: Published by Harvard University. 1897.

This little book deals with an important topic. The presentation aims to acquaint the student with the nature and use of these series and to introduce him to the theory in such a way that at each step he sees precisely the question at issue. As aids to this end the work gives a variety of illustrations of applications of these series to computations in pure and applied mathematics, a full and careful exposition of the meaning and scope of the more difficult theorems, and the use of diagrams and graphical illustrations in the proofs. We have read these chapters with much interest and heartily commend the book to our readers as a valuable supplement to the treatment given in the usual text-books on the Differential and Integral Calculus. J. M. C.

Intermediate Algebra. University Tutorial Series. By William Briggs, M. A., F. C. S., F. R. A. S., and G. H. Bryan, Sc. D., F. R. S. 375 pages. Price, $1.00. London: W. B. Clive. New York Depot: Hinds and Noble.

This is a work of more than ordinary merit. It is based on the treatise of Radhakrishnan, with such alterations and additions as were necessary to render it suitable to the wants of English and American students. The simple properties of Inequalities are treated at an early stage, the important properties of Zero and Infinity are adequately presented, and the theory of Quadratic expressions and Maxima and Minima are fully discussed. The chapters on Logarithms, Interest and Annuties are excellent in every detail. J. M. C.

Elementary and Constructional Geometry. By Edgar H. Nichols, A. B., of the Brown and Nichols School, Cambridge, Mass. Pages 138. New York: Longmans, Green & Co.

This book is very carefully written and is admirably adapted for the place it is designed to fill. The author uses the words *symparallel* and *antiparallel* for parallel lines

that have the same and the opposite directions, respectively. A proper use of the blank pages at the end of the book for a summary of facts, definitions, and principles will add greatly to the usefulness of the book. J. M. C.

The Science of Mechanics. A Critical and Historical Exposition of Its Principles. By Dr. Ernst Mach, Professor of Physics in the University of Prague. Translated from the Second German Edition by Thomas J. McCormack. With two hundred and fifty cuts and illustrations. Half morocco, gilt top, marginal analysis, exhaustive index. Price, $2.50. Chicago: The Open Court Publishing Co.

This is one of the most readable works on Mechanics that has yet come to our notice. The rigorous and rigid mathematical reasoning is interspersed by many interesting historical facts concerning the application and development of the principles under consideration, as well as giving some pleasing accounts of the first discoveries of these principles. The work is in every way worthy the highest patronage, and no difference what text-book on Mechanics may be adopted for class use, Dr. Mach's book ought to be in use in every class to supplement the work of the regular course. The book is beautifully printed and handsomely bound. B. F. F.

Elementary Mathematical Astronomy. With Examples and Examination Papers. By C. W. C. Barlow, M. A., B. Sc., Gold Medalist in Mathematics at London M. A.; Sixth Wrangler, and First Class First Division Part II. Mathematical Tripos, Cambridge, and G. H. Bryan, M. A., Sc. D., F. R. S., Smith's Prizeman, Fellow of St. Peter's College, Cambridge; Joint Author's of "Coördinate Geometry." 16mo. cloth. 442 pages. Price, $1.50. London: W. B. Clive, University Correspondence College Press; and New York: Hinds and Noble.

Nothing but words of praise can be said of this work. A somewhat careful examination leads us to pronounce it the best in the particular field it is designed to cover. The book gives a most excellent description of the methods by which the structure of Scientific Astronomy has been built up with a very small amount of mathematical knowledge. The book should be the delight of every student of Astronomy. The arrangement is good, the diagrams clear and accurate, and the whole treatment excellent. B. F. F.

The Open Court. A Monthly Magazine devoted to the Science of Religion, the Religion of Science, and the Extention of the Religious Parliament Idea. Edited by Dr. Paul Carus; T. J. McCormack, Assistant Editor; E. C. Hegeler. and Mary Carus, Associate Editors. Price, $1.00 per year in advance. The Open Court Publishing Co., Chicago, Ill.

Among the articles in the August number are the following: The Religion of Islam, by Hyacinthe Loyson; History of the People of Israel, from the Beginning of the Destruction of Jerusalem, by Dr. C. H. Corniell, Professor of Old Testament History in the University of Königsberg; and the Evolution of Evolution, by Dr. Moncure D. Conway.

B. F. F.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year in advance. Single number, 10 cents. Irvington-on-the-Hudson.

The Mathematical Gazette. Edited by F. S. Macauley, St. Paul's School, West Kensington, London. Issued three times a year, viz : in February, June, and October. Price, one shilling, net.

The June number contains an article on Spherical Geometry: I. Orthogonal Projection, by Prof. Alfred Lodge, M. A.; II. Stereographic Projection, by P. J. Heawood, M. A. Also Notes, Mathematical Notes, Examination Questions and Problems, Solutions, and Reviews and Notices. In "Notes" is an extended notice of Dr. Halsted's article on the "Non-Euclidean Geometry" which appeared in the March number of the MONTHLY.

B. F. F.

The Monist. A Quarterly Magazine devoted to the Philosophy of Science. Edited by Dr. Paul Carus ; T. J. McCormack, Assistant Editor ; E. E. Hegeler, and Mary Carus, Associate Editors. Price, $2.00 per year in advance. Single number, 50 cents. The Open Court Publishing Co., Chicago, Ill.

The following articles appeared in the January, 1897, number: The Logic of Relatives, by Chas. S. Peirce; Man as a Member of Society, Introduction, by Dr. P. Topinard; The Philosophy of Budhism, by Dr. Paul Carus; Panlogism, by E. Douglas Fawcett; The International Scientific Catalogue, and the Decimal System of Classification, by Thomas J. McCormack; and Literary Correspondence—France, by Lucien Arréat. B. F. F.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single Number, 25 cents. The American Monthly Review of Reviews Co., 13 Astor Place, New York City.

We are pleased to note that since our last issue this valuable magazine has changed its name to *The American Monthly Review of Reviews*, a more significant title than its former one.

The September number has a good deal to say about the Andrews incident and Brown University—not so much, as the editor remarks, on account of the personal interests involved in the case, as because of the far-reaching principles affecting academic life and liberty which have become matters at issue. A fair-minded and judicious estimate of President Andrews' services to Brown is given by a writer fully conversant with the facts, and the protest of the faculty is printed in full. The editorial comments on the awkwardness and needlessness of the situation are piquant and to the point.

Among the contributed articles in the September number are sketches of the three members of the new Nicaraguà Canal Commission—Admiral Walker, Capt. O. M. Carter, Corps of Engineers, U. S. A., and Prof. Lewis M. Haupt. These sketches are illustrated with portraits, and serve to convey an idea of the peculiar qualifications possessed by these gentlemen for the task to which they have been appointed by President McKinley.

B. F. F.

The Arena. An Illustrated Monthly Magazine. Edited by John Clark Ridpath, LL. D. Price, $2.50 per year in advance. Single number, 25 cents. Boston : The Arena Co.

Every true American citizen should read Dr. John Clark Ridpath's splendid paper, "The Cry of the Poor," and his "Open Letter" to President E. B. Andrews, which appear in the September number of the *Arena*. In them the Doctor has drawn a picture that appeals to every man and woman in our land who has God-given rights and privileges which, owing to the intervention of plutocratic influences, they are not allowed to enjoy.

"Why," asks the Doctor, "should the voice of the poor ever be heard rising like a wail from plantation, hamlet, and cityful ? Why should there be seen standing at the

door of the homes of the American people the gaunt spectre—Want ?" "And why," he again asks, "should we allow the voice of our teachers to be smothered by plutocratic powers ?" There may be those who sanction the conduct of Brown University in expelling Professor Andrews, but it is very evident that the editor of the *Arena* and the author of "The Bond and the Dollar" and "The True Inwardness of Wall Street" does not.

Among the other papers are "The Concentration of Wealth, its Cause and Results: Part I," by Herman E. Taubeneck; "The Multiple Standard for Money," by Eltweed Pomeroy; "The Future of the Democratic Party: A Reply," by David Overmyer; "The Author of 'The Messiah'," by B. O. Fowler; "Anticipating the Unearned Increment," by I. W. Hart; "Studies in Ultimate Society:" I. "A New Interpretation of Life," by Laurence Gronlund; II. "Individualism vs Altruism," by K. T. Takahashi; "General Weyler's Campaign," by Crittenden Marriott; the "Plaza of the Poets," "Book Reviews," and "The Editor's Evening," make up this bright and instructive number.

CORRECTIONS AND REVISIONS OF THE ARTICLE

"ON THE CIRCULAR POINTS AT INFINITY,"

MAY MONTHLY, PP. 132—145.

(P.=page ; l. $\varkappa$=$\varkappa$th line from above ; lb. $\varkappa$=$\varkappa$th line from below.)

P. 132, l. 1 of the article, read Coördinate for Coordinate ; l. 2, Cartesian for Cartesion. P. 133, (4) and (4)′ for (A) and (A)′; l. 19—21, finish parenthesis ; l. 23, = for —. P. 134, l. 2, vanishes for vanisnes ; interchange lines 14 and 15. P. 135, l. 4, bring "all true" down to l. 6 ; l. 12, add "and" after "infinity"; l. 19 and 23, coördinates for coöordinates ; l. 25, coördinates for coödinates. P. 136, l. 4, add exponent 2 to numerator ; l. 6, ρ^2 here taken equal to 1, might have been retained in the numerator. If retained, (21) p. 140 would contain ρ^4 instead of ρ^2, but this would have no effect on the final result (22). Whether ρ^2 is retained or not, (14) would have to be made homogeneous in all the coördinates involved, as well as (21), for practical uses, since this is required of all such equations. (14) can be made homogeneous by the use of the solution of (4). l. 9, $+\sin\alpha_1\sin\alpha_2$ for $-\sin\alpha_1\sin\alpha_2$; $-\varkappa_1\varkappa_2\cos C$ for $\varkappa_1\varkappa_2\cos C$; lb. 6, c for C. P. 137, l. 16, r for γ. P. 138, lb. 4, $x_2'^2$ for $x_1'^2$; lb. 5, r for γ ; lb. 8, $\cos C$ for $\cos B$; lb. 9, $x_1'^2$ for x_1^2. P. 139, l. 5, $\varkappa_3^2$ for x_3^2 and for $\varkappa_3$; l. 12, $x_3'x_1$ for x_3x_1. P. 140, lb. 2, $(x'\xi\xi')^2$ for $(x'\xi\xi)^2$. P. 141, l. 14, x' for x ; l. 17, $\varkappa_2^2u_2^2+\varkappa_3^2u_3^2$ for $\varkappa_2u_2^2+\varkappa_3u_3^2$; lb. 1, $x_2'^2$ for x_2' ; in foot note, "Nicht-Euklidische Geometrie" for "Nicht-Euclidsche Geometry." P. 142, l. 9, $-iA$ for $-iB$; l. 11, *two lines* for *the lines* ; l. 13 and 14, x and y might be interchanged, though this is not necessary ; the other angle between the two lines would be given ; l. 15, The double ratio of these is : Taking them in the order named, using etc.; l. 18, s for 5 ; l. 19, $+s\lambda'$ for $+s\lambda$. P. 144, l. 11, $\tan\psi$ for ψ ; slopes for tangents would be better ; l. 12, it is necessary and sufficient that the purely imaginary part of x should become indefinitely great ; l. 18, the German word "quadrupel" is here appropriated ; l. 23, ± 1 for $\pm l$; l. 27, is for in ; in foot note, * for †. P. 145, l. 4, $\Sigma xx.\Sigma x'x'$ for $\Sigma xx'.\Sigma x'x$ in numerator and denominator ; $(\Sigma xx')^2$ for $\Sigma xx'$ under radical in denominator ; l. 5, two points for the points ; l. 7, Σxx for $\Sigma xx'$.

G. B. M. ZERR — WILLIAM E. HEAL — HON. JOSIAH H. DRUMMOND
JOHN M. ARNOLD — O. W. ANTHONY

THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. IV. OCTOBER, 1897. No. 10.

SOPHUS LIE'S TRANFORMATION GROUPS.

A SERIES OF ELEMENTARY, EXPOSITORY ARTICLES.

By EDGAR ODELL LOVETT, Princeton University.

I.

1. Without entering unnecessarily into definitions which will occur more properly later, the following paragraph may serve for purposes of orientation. Among the most important notions of modern pure mathematics are the idea of a *group* and its associated notions *transformation*, *substitution*, *invariant* and *differential invariant*. Groups fall naturally and historically into two classes, *discontinuous* and *continuous*. The former are usually called *substitution groups* and are not infrequently referred to as GALOIS' groups ; the latter are known as *continuous transformation groups* and may with propriety be called LIE groups. Substitution groups find their greatest usefulness in the theory of algebraic equations, with a limited range of application to geometry ; transformation groups play a similar rôle in the theory of differential equations, with a wide application to geometry and mechanics. The idea of a substitution group in its modern signification and in its relation to the theory of algebraic equations is due to GALOIS ; LIE, after having modified and extended the idea of a substitution group, introduced the new notion into the domain of analysis and geometry and thus created his theory of transformation groups.

The great fruitfulness and remarkable simplicity of LIE's theories are their most striking characteristics. Because of their manifold applications, a thorough and systematic study of the fundamental properties of continuous groups is cer-

tain to yield the reader a liberal education in mathematics ; in addition to a knowledge of the technicalities of the theory of groups, there is gained at the same time a properly proportioned perspective of the many fields of the science brought into one domain. The group idea is a unifying principle which tends to reduce the various and in many cases apparently heterogeneous subjects of mathematics into a homogeneous body of doctrine.

It is the purpose of these notes to present some of the more elementary theorems of LIE's theories and to call attention to a few of the many applications to geometry and differential equations. The material has been drawn from the numerous published treatises* and memoirs of LIE and from his lectures delivered at the University of Leipsic in 1895 and '96. In order to an intelligent perusal of the sequel no more is required than a familiarity with the facts and processes of elementary mathematics including the simpler operations of the differential and integral calculus.

2. The simplest LIE groups are those of one parameter ; however, before proceeding to the fundamental theorems of the theory of groups of one parameter a few examples already familiar to the reader of analytical geometry as transformations of coördinates will be useful in introducing the notions.

For the sake of simplicity let the study be made in the plane. Consider the plane as a manifoldness of points, *i. e.* as a space whose space element is a point. Since it takes two independent coördinates to fix the position of a point in the plane we may say that there are ∞^2 points† in the plane or, what amounts to the same thing, that the plane is a two-dimensional space if the point is its space element. Consider the ensemble of all points of the plane and suppose that this aggregate be moved a given distance in a given direction. By this *translation* every point in the plane will be carried into the position of one of the others. In order to represent this analytically, let us suppose that the x-axis of a Cartesian coördinate system lies in the direction of the translation and that the distance through which all the points of the plane are moved is a, then the point (x, y) is carried over into the point

$$x_1 = x + a, \quad y_1 = y.$$

The segment a can be given all values from $-\infty$ to $+\infty$, and if a be varied in this manner we obtain ∞^1 translations in one and the same direction or in its opposite direction.

*A list of these treatises is to be found in the June (1897) number of the Bulletin of the American Mathematical Society or in Teubner's catalogue. The reader who desires to prosecute the study of the subject further than the scope of these notes will find the following order of attack on Lie's published works the most satisfactory: 1° Lectures on Differential Equations with Known Infinitesimal Transformations; 2° Lectures on Continuous Groups; 3° Geometry of Contact Transformations; 4° Theory of Transformation Groups, the three volumes of this treatise in their order.

†This notation is very convenient. Its general form is -If a configuration depends on n independent parameters, of which none is superfluous, the configuration assumes ∞^n positions if the parameters are allowed to vary from $-\infty$ to $+\infty$. So, for example, there are ∞^1 points on a line, ∞^2 in the plane, ∞^3 in space, since the position of the point depends on one, two, or three parameters, respectively. Similarly there are ∞^3 circles in the plane, ∞^4 straight lines in space, ∞^5 conics in the plane, and so on. The symbol ∞^n in this connection is read "n-ply infinite number of."

Suppose now that two of these translations be carried out in succession, the first through the distance a changes the point (x, y) into the point

$$x_1 = x + a, \quad y_1 = y,$$

and the second through the distance a_1 carries the new point (x_1, y_1) into the position

$$x_2 = x_1 + a_1, \quad y_2 = y_1,$$

which together with (x, y) and (x_1, y_1) lies on a parallel to the x-axis. Now it is clear geometrically, that the passage from the initial position (x, y) to the final position (x_2, y_2) can be effected by a single translation through the distance $a + a_1$, and in fact simultaneously for all points of the plane. This also appears analytically from the fact that the elimination of the intermediate position (x_1, y_1) from the above equation gives

$$x_2 = x + a + a_1, \quad y_2 = y.$$

This very simple result may be formulated in the following manner:

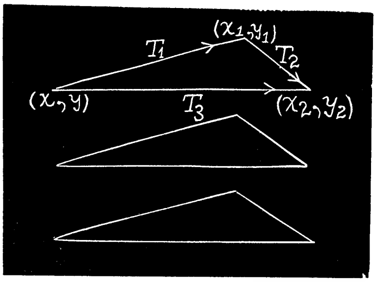
The successive application of two translations of the family of ∞^1 *translations*

$$x_1 = x + a, \quad y_1 = y$$

is equivalent to a translation belonging to the same family.

For this reason the family is called a *group of translations.* It contains one arbitrary parameter a and hence ∞^1 translations; accordingly it is said to be a *one-parameter group.*

3. So far the translations have been limited in direction; let us now consider all translations in the plane. As in the preceding case let all the points of the plane be moved through the same distance a and in the same direction α; if a and α be given all possible values we obtain a family of ∞^2 translations which includes the preceding family as a particular case. Any one of the translations of the family changes the point (x, y) into the point



$$x_1 = x + a, \quad y_1 = y + b,$$

where a and b are two arbitrary values but remain the same for all points of the plane. If a translation, T_1, carry the point (x, y) into the position of the point (x_1, y_1), and a second translation, T_2, carry the point (x_1, y_1) over to (x_2, y_2), it is clear geometrically that the point (x, y) could have been carried directly to the position (x_2, y_2) by a single translation, T_3. The length and direction of

this third translation T_3, equivalent to the successive application of T_1 and T_2, is found by constructing the third side of the triangle formed by the translations T_1 and T_2, or in kinematical parlance, by taking the geometric sum of T_1 and T_2. This result appears analytically by eliminating x_1, y_1 from the equations representing the translations

$$T_1, \quad x_1 = x + a, \quad y_1 = y + b\;;$$

$$T_2, \quad x_2 = x_1 + a_1, \quad y_2 = y_1 + b_1\;;$$

this elimination yields the equation

$$T_3, \quad x_2 = x + a + a_1, \quad y_2 = y + b + b_1,$$

which is of the same form as the equations representing T_1 and T_2 and hence belongs to the same family as T_1 and T_2 ; therefore we conclude that

The successive application of any two translations of the family of all translations of the plane

$$x_1 = x + a, \quad y_1 = y + b$$

is equivalent to a single translation belonging to the same family.

Because of the possession of this remarkable property* the family of all translations of the plane is called a *group of translations*. The *group* contains two arbitrary constants a and b, *i. e.* it has ∞^2 different translations ; for this reason the group of all translations in the plane is called a *two-parameter* group.

4. In order to present simple concrete examples illustrative of several other fundamental notions let us return to the family of all translations parallel to the x-axis

$$x_1 = x + a, \quad y_1 = y\;; \tag{1}$$

among these ∞^1 translations there is one to be noted, namely that one for which

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*It is easy to see that this property of the equivalence of the successive applications of any two transformations of a family of transformations to a third transformation belonging to the same family is a remarkable one, peculiar to certain families, and not common to all. For example, the equations,

$$x_1 = a - x, \quad y_1 = y,$$

represent a family of ∞^1 transformations, which may be readily constructed geometrically, but a transformation S_1 changing (x, y) into

$$x_1 = a - x, \quad y_1 = y,$$

followed by S_2, changing (x_1, y_1) into

$$x_2 = a_1 - x_1, \quad y_2 = y_1,$$

produces, by the elimination of (x_1, y_1) from these equations, the equations

$$x_2 = (a_1 - a) + x, \quad y_2 = y,$$

which represents the transformation S_3 equivalent to the successive application of S_1 and S_2. But the x of the original family is equal to a constant minus the old x, while in S_3 the new x is equal to a constant plus the old x, hence S_3 does not belong to the same family as S_1 and S_2. The ∞^1 transformations represented by the above equation then do not form a Lie group.

$a=0$, *i. e.* a translation through the distance zero. By this translation all points of the plane remain at rest, and strictly speaking the term translation is no longer allowable. If, for the sake of continuity, the term translation is to be retained as applicable to this case also, then the translation by which every point is changed into itself is called the *identical translation*. It is to be further remarked that for every translation of this group there is a translation of the group which, carried out after the former, cancels its effect. Thus the successive application of the translations corresponding to $+a$ and to $-a$ respectively is equivalent to the translation $a-a=0$, that is, to the identical transformation. For this reason the two translations are said to be inverse.

If we put a equal to an infinitely small constant ∂t, we obtain an *infinitesimal translation*, which gives to all points of the plane only an infinitely small motion

$$x_1=x+\partial t, \quad y_1=y.$$

By this translation the coördinates x, y receive infinitely small increments

$$\partial x=\partial t, \quad \partial y=0,$$

and if the *infinitesimal translation* be carried out n times successively, the point (x, y) is changed into

$$x_1=x+n\partial t, \quad y_1=y:$$

if the infinitesimal translation be repeated an infinite number of times, or, what comes to the same thing, if n becomes infinite, then $n\partial t$ is equal to some finite quantity a and we have again a finite translation

$$x_1=x+a, \quad y_1=y.$$

We shall find later on that a one-parameter group contains but one infinitesimal transformation.

Suppose that we operate on a definite point (x_0, y_0) with all translations of the one-parameter group (1); the point will take ∞^1 different positions:

$$x=x_0+a, \quad y=y_0,$$

the aggregate of which is a parallel to the x-axis. This line, the locus of all the points into which a definite point is changed by operating on it with all the translations of the group, is called the *path curve of the point*, or *path curve of the one-parameter group*. There are altogether ∞^1 path curves of the group (1) consisting of the family of straight lines parallel to the x-axis.

Any translation of the group carries any one of the path curves, as a whole, forward in its own direction a distance a; *i. e.* the path curve as a whole remains at rest. The path curves are invariant by all the translations of the

group. In addition to the line at infinity and the path curves, there is no other *invariant curve* by this one-parameter group, *i. e.* no other curve all of whose points are changed into points of the same curve by all the transformations of the group.

If a function of (x, y), $F(x, y)$ is to be invariant by the group

$$x_1=x+a, \quad y_1=y,$$

we must have $F(x_1, y_1)\equiv F(x+a, y)=F(x, y)$, for all values of a. In order to determine the function F we need only to take an infinitesimal value for a, and carry out the infinitesimal translation $x_1=x+\partial t$, $y_1=y$.* Taylor's series gives

$$F(x, y)+\frac{a}{1}\frac{\partial F(x, y)}{\partial x}+\frac{a^2}{1.2}\frac{\partial^2 F(x, y)}{\partial x^2}+\ldots\ldots=F(x, y),$$

or cancelling $F(x, y)$ from each side and neglecting terms of the second order

$$\frac{\partial F(x, y)}{\partial x}=0,$$

that is, F does not contain x and is a function of y alone. Hence every function $F(y)$ is an *invariant function* by the one-parameter group (1). An *invariant function* equated to a constant gives an *invariant equation*, which represents one or more *path curves* of the group.

The reader may find it interesting to verify the group property for the following families and to determine the path curves and forms of invariant functions :

1° Rotations about a fixed point $\begin{cases} x_1=x\cos\alpha-y\sin\alpha, \\ y_1=x\sin\alpha+y\cos\alpha\,; \end{cases}$

2° The affine transformations $x_1=ma, \ y_1=y$;

3° The perspective transformations $x_1=ax, \ y_1=ay$;

4° The transformations $x_1=ax, \ y_1=y/a$;

5° The group of all Euclidean motions in ordinary space

$$x_1=a_1x+a_2y+a_3z+a_0,$$

$$y_1=b_1x+b_2y+b_3z+b_0,$$

$$z_1=c_1x+c_2y+c_3z+c_0.$$

The University of Chicago, 10 September, 1897.

[To be Continued.]

*In order that a function, equation or curve be invariant by all of the finite transformations of a one-parameter group, it is necessary and sufficient that the function, equation or curve be invariant by the infinitesimal transformation of the group. This theorem will be proved in the sequel.

ON A SOLUTION OF THE GENERAL BIQUADRATIC EQUATION.

By A. C. BURNHAM, Professor of Mathematics, University of Illinois, Urbana, Illinois.

Very often in mathematical work does one wish to write out without waste of time the value of the unknown in a given biquadratic equation. Nowhere in text-books or mathematical writings do I find the solution to a biquadratic given in such form that one by merely substituting in a formula may get the roots. I have found the formula here given convenient and I do not know that the formula or this particular method of getting the result has ever before been published.

Let the general biquadratic be

$$x^4+a_1x^3+a_2x^2+a_3x+a_4=0,$$

and let the roots be a, b, c, d. Then follow, as is well known,

$$\begin{aligned} a+b+c+d&=-a_1, \\ ab+ac+ad+bc+bd+cd&=a_2, \\ abc+abd+acd+bcd&=-a_3, \\ abcd&=a_4. \end{aligned}$$

Now let

$$\left.\begin{aligned} z_1&=ab+cd \\ z_2&=ac+bd \\ z_3&=ad+bc \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathrm{I}.$$

Then it follows that,

$$\begin{aligned} z_1+z_2+z_3&=a_2, \\ z_1z_2+z_1z_3+z_2z_3&=(ab+cd)(ac+bd)+(\quad)(\quad)+(\quad)(\quad) \\ &=a^2bc+ab^2d+c^2ad+cbd^2+\ldots\ldots+\ldots\ldots \\ &=\Sigma a^2bc=a_1a_3-4a_4, \end{aligned}$$

and

$$\begin{aligned} z_1z_2z_3&=(ab+cd)(ac+bd)(ad+bc) \\ &=\Sigma a^3bcd+\Sigma a^2b^2c^2 \\ &=a_3^2+a_1^2a_4-4a_2a_4. \end{aligned}$$

Then z_1, z_2, z_3 are therefore the roots of the reducing cubic:

$$z^3-a_2z^2+(a_1a_3-4a_4)z-(a_3^2+a_1^2a_4-4a_2a_4)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \mathrm{II}.$$

Now from I we have

$$z_1^2 = a^2b^2 + c^2d^2 + 2abcd$$
$$= a^2b^2 + c^2d^2 + 2a_4,$$
$$\therefore\ z_1^2 - 4a_4 = a^2b^2 + c^2d^2 - 2abcd = (ab-cd)^2,$$
$$\therefore\ \sqrt{z_1^2 - 4a_4} = ab - cd \ \ldots\ldots\ldots\ldots (a),$$
$$\text{but } z_1 = ab + cd \ \ldots\ldots\ldots\ldots (b).$$

Therefore by adding (a) and (b),

$$ab = \tfrac{1}{2}\{z_1 + \sqrt{z_1^2 - 4a_4}\} \ \ldots\ldots\ldots\ldots (c),$$

and by subtracting (a) from (b) we have

$$cd = \tfrac{1}{2}(z_1 - \sqrt{z_1^2 - 4a_4}) \ \ldots\ldots\ldots\ldots (d).$$

In the same manner we get

$$ac = \tfrac{1}{2}(z_2 + \sqrt{z_2^2 - 4a_4}) \ \ldots\ldots\ldots\ldots (e),$$
$$bd = \tfrac{1}{2}(z_2 - \sqrt{z_2^2 - 4a_4}) \ \ldots\ldots\ldots\ldots (f),$$
$$ad = \tfrac{1}{2}(z_3 + \sqrt{z_3^2 - 4a_4}) \ \ldots\ldots\ldots\ldots (g),$$
$$bc = \tfrac{1}{2}(z_3 - \sqrt{z_3^2 - 4a_4}) \ \ldots\ldots\ldots\ldots (h).$$

But $ab + ac + ad = a(b + c + d)$

$= (-a_1 - a)a$, since $b + c + d = -a_1 - a$

$= -a^2 - a_1 a.$

Also $ab + ac + ad = \frac{1}{2}\{z_1 + z_2 + z_3 + \sqrt{z_1^2 - 4a_4} + \sqrt{z_2^2 - 4a_4} + \sqrt{z_3^2 - 4a_4}\}$ from (c), (e), and (g). Therefore,

$$a^2 + a_1 a + \tfrac{1}{2}\{a_2 + \sqrt{z_1^2 - 4a_4} + \sqrt{z_2^2 - 4a_4} + \sqrt{z_3^2 - 4a_4}\} = 0,$$

which is a biquadratic equation giving the value of one root a, *i. e.*

$$a = \frac{-a_1 \pm \sqrt{a_2 - 2\{a_2 + \sqrt{z_1^2 - 4a_4} + \sqrt{z_2^2 - 4a_4} + \sqrt{z_3^2 - 4a_4}\}}}{2} \ \ldots\ldots$$

The four roots are, therefore,

$$\left.\begin{matrix} a \\ b \\ c \\ d \end{matrix}\right\} = \tfrac{1}{2}\left\{-a_1 \pm \sqrt{a_1^2 - 2\{a_2 \pm \sqrt{z_1^2 - 4a_4} \pm \sqrt{z_2^2 - 4a_4} \pm \sqrt{z_3^2 - 4a_4}} \right. \ \ldots\ldots \mathrm{III},$$

where the sequence of signs under the main radical, as can be seen from formulæ (c) to (h), is

for a,	$+$	$+$	$+$
for b,	$+$	$-$	$-$
for c,	$-$	$+$	$-$
for d,	$-$	$-$	$+$

For the z_1, z_2, z_3 in this solution III must be substituted the roots of the cubic II.

EXAMPLE. As an example take the biquadratic

$$x^4-x^3-7x^2+x+6=0.$$

Here we have,

$$a_1=-1, \quad a_3=1,$$
$$a_2=-7, \quad a_4=6,$$

from which the cubic becomes $z^3+7z^2-25z-175=0$, of which the roots are 5, -7, and -5. Thus the roots of the biquadratic are

$$\tfrac{1}{2}\{1\pm\sqrt{1-2\{-7\pm1\pm5\pm1\}},$$

or 1, -1, -2, 3, which are seen to be correct.

Care must be exercised that the proper sign before the main radical is taken.

Urbana, Ill., October 9, 1897.

EQUATION OF PAYMENTS.

By J. A. CALDERHEAD, A. B., Professor of Mathematics, Curry University, Pittsburg, Pennsylvania.

Let it be required to find the equated time of two payments, P and P_1, due at the end of t and t_1 years respectively, and r being the rate of interest.

Represent the equated time by x when $t>t_1$.

I. BY SIMPLE INTEREST.

1st Method. The discount on P for $(t-x)$ years must equal the interest on P_1 for $(x-t_1)$ years.

$\dfrac{P(t-x)r}{1+(t-x)r}=$discount on P due $(t-x)$ years hence.

$P_1(x-t_1)r=$interest on P_1 for $(x-t_1)$ years.

$$\therefore \frac{P(t-x)r}{1+(t-x)r}=P_1(x-t_1)r.$$

$$\therefore x = \frac{1}{P_1 r}\Big(P + P_1 + P_1 rt + P_1 rt_1$$

$$\pm \sqrt{P^2 + P_1^2 + P_1^2 r^2 t^2 + P_1^2 r^2 t_1^2 + 2PP_1 + PP_1 rt + 2PP_1 rt_1 + P_1^2 rt}\Big) \text{ .... } (1).$$

2nd Method.

$\dfrac{P}{1+rt}$ = present worth of P due t years hence.

$\dfrac{P_1}{1+rt_1}$ = present worth of P_1 due t_1 years hence.

$\dfrac{P+P_1}{1+rx}$ = present worth of $P+P_1$ due x years hence.

$$\therefore \text{Suppose } \frac{P}{1+rt} + \frac{P_1}{1+rt_1} = \frac{P+P_1}{1+rx};$$

$$\text{then } x = \frac{Pt + P_1 t_1 + Prtt_1 + P_1 rtt_1}{P + P + Prt_1 + P_1 rt} \text{ ....} (2).$$

Since (2) differs from (1), the sum of the present worths of P and P_1 due in t and t_1 years respectively, at simple interest, is not equal to the present worth of $P+P_1$ due at the equated time; hence, the second method is not correct when we compute by simple interest.

II. By Compound Interest.

1st Method.

$P\left[1 - \dfrac{1}{(1+r)^{t-x}}\right]$ = discount on P for $(t-x)$ years.

$P_1[(1+r)^{x-t_1} - 1]$ = interest on P_1 for $(x-t_1)$ years.

$$\therefore P\left[1 - \frac{1}{(1+r)^{t-x}}\right] = P_1[(1+r)^{x-t_1} - 1].$$

$$\therefore x = \frac{\log(P+P_1)[(1+r)^t(1+r)^{t_1}] - \log[P(1+r)^{t_1} + P_1(1+r)^t]}{\log(1+r)} \text{ ......} (3).$$

2nd Method.

$\dfrac{P}{(1+r)^t}$ = present worth of P due t years hence.

$\dfrac{P_1}{(1+r)^{t_1}}$ = present worth of P_1 due t_1 years hence.

$\dfrac{P+P_1}{(1+r)^x}$ = present worth of $P+P_1$ due x years hence.

$$\therefore \text{Suppose } \frac{P}{(1+r)^t} + \frac{P_1}{(1+r)^{t_1}} = \frac{P+P_1}{(1+r)^x}.$$

Then $x=\dfrac{\log(P+P_1)[(1+r)^t(1+r)^{t_1}-\log[P(1+r)^{t_1}+P_1(1+r)^t]}{\log(1+r)}$ ....(4).

But (4) and (3) being identical, either method may be used when compound interest is considered. The first, or correct, method by simple interest becomes very complicated when more than two payments are considered ; yet when we recall the fact that equation of payments is a subject of no practical importance, making approximate methods less desirous, it matters little how complicated the method may be if it is correct in theory.

The following method, which is found in most arithmetics is very often not much better than a good guess. A review of the solution will, at once, show the erroneousness of the method.

$P(t-x)r$ = interest on P for $(t-x)$ years.

$P_1(x-t_1)r$ = interest on P_1 for $(x-t_1)$ years.

$P(t-x)r=P_1(x-t_1)r.$

$\therefore\ x=\dfrac{Pt+P_1t_1}{P+P_1}.$

III. BY ANNUAL INTEREST.

$\dfrac{Pr[(t-x)+\frac{1}{2}r(t-x)(t-x-1)]}{1+r[(t-x)+\frac{1}{2}r(t-x)(t-x-1)]}$ = discount on P for $(t-x)$ years.

$P_1r[(x-t_1)+\frac{1}{2}r(x-t_1)(x-t_1-1)]$ = interest on P_1 for $(x-t_1)$ years.

$\therefore\ \dfrac{Pr[(t-x)+\frac{1}{2}r(t-x)(t-x-1)]}{1+r[(t-x)+\frac{1}{2}r(t-x)(t-x-1)]}=P_1r[(x-t_1)+\frac{1}{2}r(x-t_1)(x-t_1-1)]$ .....(5).

From (5) x, the equated time, can be found.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED. A. M., (Princeton) ; Ph. D., (Johns Hopkins) ; Member of the London Mathematical Society ; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from the August-September Number.]

PROPOSITION XXVIII. *If two straights AX, BX (produced from any-sized straight AB toward the same parts, the first under an acute angle, and the other perpendicularly) mutually approach each other ever more without any certain limit, save at their infinite production ; I say all angles (Fig. 33.) at any points L, H, D of AX, from which are dropped to the straight BX perpendiculars LK, HK, DK,*

first will all be obtuse toward the parts of the point A, secondly will be ever less, the more distant from this point A, and finally the angles more and more distant from this same point A ever more without any certain limit approach to equality with a right angle.

Demonstratur. The first part follows indeed from Corollary I to Proposition XIII. The second part however is proved thus: For the two angles together at LK toward the base AB are greater (from Corollary to Proposition XVI.) than the two internal and opposite angles together at HK toward the same base AB.

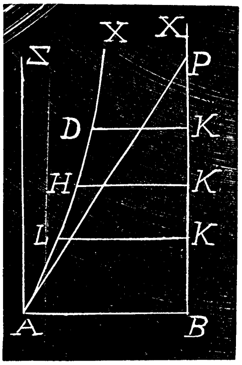


Fig. 33.

But the angles at each point K toward the base AB are equal to each other, as being right. Therefore the obtuse angle at L toward the base AB is greater than the obtuse angle at H toward the same base AB.

In like manner is shown that the aforesaid obtuse angle at H is greater than the obtuse angle at the point D.

And thus ever, proceeding toward the points X.

Finally the third part requires a longer disquisition. If therefore it can be done, let there be assigned (Fig. 34.) a certain angle MNC, than which is always greater, or anyhow not less, the excess of any of the aforesaid obtuse angles above a right angle. It follows (from Proposition XXI.) that the sides NM, NC comprehending that angle MNC can be so produced that the perpendicular MC from a certain point M of MN let fall upon NC may be greater (even in the hypothesis of acute angle) than any assigned finite length, as for instance the aforesaid base AB.

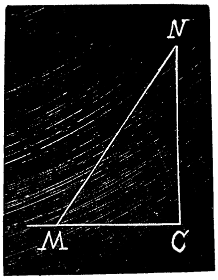


Fig. 34.

This standing ; assume in BX (Fig. 35.) a certain BT equal to CN, and erect from the point T toward AX the perpendicular TS, which obviously (from Scholion to Proposition XXIV.) meets AX in a certain point S. Then from the point S let fall to AB the perpendicular SQ.

This falls (because of Euclid I. 17.) toward the parts of the acute angle between the points A and B. Again, acute will be the angle QST in the quadrilateral $QSTB$, since the remaining three angles are right ; else (against Proposition V. and Proposition VI.) we come upon the hypothesis either of right angle or of obtuse angle.

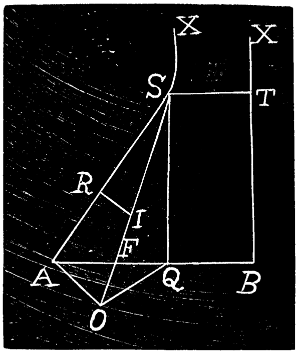


Fig. 35.

Hence the straight SQ will be greater (from Corollary I. to Proposition III.) than the straight BT,

or CN ; and again the angle ASQ will be greater than the excess by which the obtuse angle AST exceeds a right angle, and thus greater than the angle MNC. Draw therefore a certain SF cutting AQ in F and making with SA an angle equal to MNC. Then from the point A draw to SF produced the perpendicular AO. The point O falls (from Euclid I. 17.) below the point F, since the angle AFS (by Euclid I. 16.) is obtuse.

Finally, however ; since FS is greater (by Euclid I. 19.) than QS and so much greater than BT or CN, assume in FS the piece IS equal to CN, and from the point I erect to FS the perpendicular IR meeting AS in the point R.

But the point R falls between the points A and S : for if it fell on any point of AF, we would have in the same triangle (against Euclid I. 17.) two angles greater than two right angles, since the angle at the point F toward the parts of the point A has already been shown obtuse.

After so much preparation thus I conclude. Since in the quadrilateral $AOIR$ the angles at the points O and I are right, and the angle at the point A (by Euclid I. 17.) is acute because of the right angle AOS, and again the angle IRA (by Euclid I. 16.) is obtuse, since the angle RIS is right : the consequence finally is (by Corollary II. to Proposition III.) that the side AO is greater than the side IR.

But (OQ joined) the side AQ is greater (by Euclid I. 19.) than the side AQ, because of the obtuse angle at O, since the angle AOS was made right.

Therefore the straight AQ will be much greater than the straight IR, or (by Euclid I. 26.) than the straight MC, and so much greater than the straight AB, the part than the whole ; which is absurd.

Therefore it is not posssible to assign any one angle MNC, than which always is greater, or anyhow not less, the excess of each of the aforesaid obtuse angles above a right angle.

Wherefore those obtuse angles, more and more distant from this point A, ever more without any certain limit approach to equality with a right angle.

Quod erat postremo loco demonstrandum.

Corollary. But this standing, which in the last case was demonstrated, it manifestly follows that those straights AX, BX, produced infinitely will finally have, either in two distinct points, or in one same point X infinitely distant, a common perpendicular.

But again, that this common perpendicular cannot be had in two distinct points flows manifestly from this, because otherwise (by Corollary II. to Proposition XXIII.) those straights would thence begin mutually to separate, and so not meet each other at an infinite distance ; so that also (against the express supposition) they would not mutually approach each other without any certain limit ever more toward those parts.

So they must have the common perpendicular in one same point X infinitely distant.

[To be Continued.]

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from June-July Number.]

XLVI. Fig. 27.
$ABLN$ is equivalent to $ABMK$ is equivalent to $ACIK$.
$NLFH=ABPO$ is equivalent to $BEDC$.
$\therefore$ $ABFH$ is equivalent to $ACIK+BEDC$.

XLVII. Fig. 27.
$ABLN$ is equivalent to $ACIK$.
$NLPO$ is equivalent to $STER$ is equivalent to $MTERC+QFD$.
$OPLH$ is equivalent to $REFH$ is equivalent to $REFQ+MBT$.
$\therefore$ $ABFH$ is equivalent to $ACIK+BEDC$.

XLVIII. Fig. 27.
$AVUH$ is equivalent to $2ACH$ is equivalent to $ACIK$.
$VBFU$ is equivalent to $2CBF$ is equivalent to $BEDC$.
$\therefore$ $ABFH$ is equivalent to $ACIK+BEDC$.
Wipper.

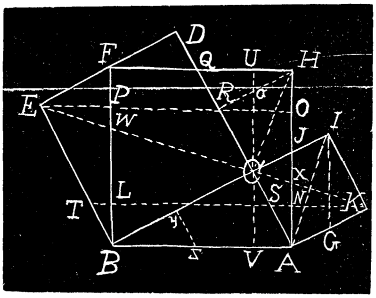


Fig. 27.

XLIX. Fig. 27.
$ABWX$, the half of $ABFH$, is equivalent to $ABC+CBW+CXA$.
But $ABC=BEF$ (is equivalent to $BWE+AXK$).
$\therefore$ $ABWX$ is equivalent to $CBE+CAK$.
$\therefore$ $ABFH$ is equivalent to $ACIK+BEDC$.

L. Fig. 27.
$Byz=FDQ$. $AzyC=AJIK$. $ARH=BEF$. $HRQ=ACJ$.
$\therefore$ $ABFH$ is equivalent to $ACIK+BEDC$.

LI. Fig. 27.
$ABC=BEF$. $CRa=FDQ$. $HRQ=IKG$. $HJCa$ is equivalent to $IGAJ$.
$\therefore$ $ABFH$ is equivalent to $ACIK+BEDC$.

That $HJCa$ is equivalent to $IGAJ$ is evident for the following reasons: $\triangle ACH$ is equivalent to $\triangle ACI$, having the same base, and equal altitudes.

Hence, subtracting $\triangle ACJ$, which is common to both, we have $\triangle CJH$ is equivalent to $\triangle AJI$.

$\therefore$ $HJCa$ is equivalent to $IGAJ$.

LII. Fig. 28.

$ABC = BEF$. $HRQ = ACJ$. $ARH = HKA$ is equivalent to $AKIJ + FDQ$.

$\therefore$ $ABFH$ is equivalent to $ACIK + BEDC$.

LIII. Fig. 28.

$AMNH$ is equivalent to $ACLH$ is equivalent to $ACIK$.

So, $MBFN$ is equivalent to $BEDC$.

$\therefore$ $ABFH$ is equivalent to $ACIK + BEDC$.

Wipper.

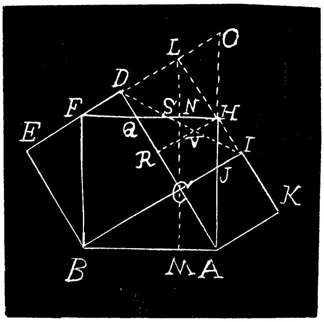


Fig. 28.

LIV. Fig. 28.

$CLOJ$ is equivalent to $CLHA$ is equivalent to $ACIK$.

$BFLC$ is equivalent to $BEDC$.

But $ABFH$ is equivalent to $BFOJ$.

$\therefore$ $ABFH$ is equivalent to $ACIK + BEDC$.

Hoffmann, 1800.

LV. Fig. 28.

$ABFH + BEF + FLH + HKA$ is equivalent to $ACIK + BEDC + ABC + CIL + CLD$.

$\therefore$ $ABFH$ is equivalent to $ACIK + BEDC$.

LVI. Fig. 28.

$ABC = BEF$. $ICD = AKH$ is equivalent to $AKIJ + FDQ$.

$SVH = SQD$, and $VHT = IJT$.

$\therefore$ By properly combining and substituting, $ABFH$ is equivalent to $ACIK + BEDC$.

LVII. Fig. 28.

$RDLH = ACIK$. $ARH = BEF$. $ABC = HFL$.

$\therefore$ $ABFH$ is equivalent to $ACIK + BEDC$.

[To be Continued.]

EUCLIDEAN GEOMETRY WITHOUT DISPUTED AXIOMS.

By G. I. HOPKINS, Instructor in Mathematics and Physics in High School, Manchester, New Hampshire.

(*a*)

PROPOSITION I. *If two straight lines in the same plane be perpendicular to the same straight line they are parallel.*

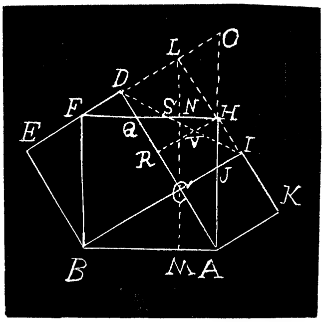
Prove by Axiom 11, and I, 27.*

*These and the subsequent numbers refer to the Book and Proposition in Todhunter's Euclid.

LII. Fig. 28.

$ABC=BEF$. $HRQ=ACJ$. $ARH=HKA$ is equivalent to $AKIJ+FDQ$.

$\therefore$ $ABFH$ is equivalent to $ACIK+BEDC$.



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[To be Continued.]

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Prove by Axiom 11, and I, 27.*

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(*b*)

PROPOSITION II. *From or through a given point in a straight line only one perpendicular to that line can be drawn in the same plane.*

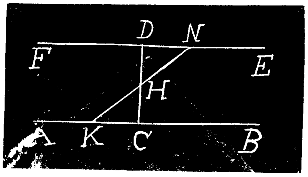
PROOF. If there could be two, there would be two unequal right angles, which is impossible by Axiom 11.

(*c*)

PROPOSITION III. *If two parallel straight lines be joined by a common perpendicular, any straight line which bisects the perpendicular and meets the two parallels is itself bisected by the perpendicular.*

Let AB be a straight line. Take any point in it as C and erect the perpendicular CD (I, XI). At D erect the perpendicular DE (I, 11) and extend it to F (Postulate 2). Then FE is parallel to AB (a).

Now bisect DC in H, (I, 10), take any point in AC as K and join KH, (Postulate 1). On DE cut off DN equal to KC, (I, 2), and join HN, (Postulate 1). Therefore the two triangles KCH and DHN are equal to each other (I, IV). Therefore KH equals HN. Again, since the two triangles KCH and DHN are equal, the angle DHN equals the angle KHC, being homologous angles. The angles KHC and KDH are together equal to two right angles (I, 13). Therefore since the angle DHN equals the angle KHC, the angles DHN and KHD are together equal to two right angles, and therefore KH and HN form one and the same straight line (I, 14). Therefore, since K is any point in AB, any straight line which bisects the perpendicular joining two parallel straight lines is bisected by the perpendicular.

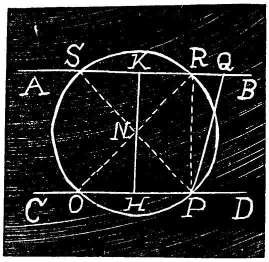


COROLLARY. If two parallel straight lines be joined by a common perpendicular, any straight line meeting the parallels and bisecting the perpendicular cuts off equal distances on the parallels on opposite sides of the perpendicular.

(*d*)

PROPOSITION IV. *If a straight line is perpendicular to one of two parallel lines it is perpendicular to the other also.*

PROOF. Let CD be a straight line. Then from any point in it as H draw HK perpendicular to CD, and in the same manner draw AB perpendicular to KH (I, 11). Then AB and CD are parallel (a). Take any point in one of the parallels as P in CD and suppose PQ be drawn perpendicular to CD. Then will PQ be perpendicular to AB also. For cut off $HO=HP$ (I, 2), bisect HK at N (I, 10), and draw PS and OR through N. Then $NO=NP$ (I, 4). But $SN=NP$ and $NO=NR$ (c). Therefore $NS=NO=NP=NR$ (Axiom 1). Therefore, similarly, $OH=HP=KR=SK$ (c, Corollary). With N as a center and NO as a radius describe a circle (Postulate 3). The circumference of this circle will obviously pass through the points O, P, R, and S. Draw PR. The angle NHO is greater than the an-

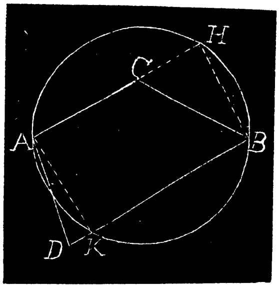


gle NPH (I, 16), therefore the angle NHP is greater than the angle NPH, and therefore NP is greater than NH (I, 19). Therefore the circumference of this circle will intersect the two parallel lines in the points O, P, R, and S. The angle OPR is a right angle (III, 31), and therefore RP is perpendicular to CD. But QP is by hypothesis perpendicular to CD, therefore PQ and PR cannot form two separate lines (b). Therefore PQ, if properly drawn must be identical with PR. But the angle SRP is a right angle (III, 31) and therefore PQ is perpendicular to AB. Q. E. D.

(e)

PROPOSITION V. *If the vertex of an angle subtended by the diameter of a circle is between the center and circumference, the angle is greater than a right angle; and if the vertex is without the circle the angle is less than a right angle.*

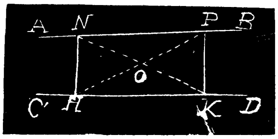
PROOF. Let AKH be a circle, AB a diameter of that circle, and let it subtend the two angles ACB and D, the vertex of the former being within, and that of the latter without, the circle. Extend AC to the circumference at point H, and join HB and KA (Postulate 1). Therefore the angles H and AKB are right angles (III, 31). Therefore the angle ACB is greater than angle H and angle D is less than angle AKB (I, 16).



(f)

PROPOSITION VI. *If two parallel straight lines be joined by two common perpendiculars, these two perpendiculars are equal to each other.*

PROOF. Let AB and CD be two parallel straight lines and let NH and PK be perpendicular to CD, then are they also perpendicular to AB (d). Join NK and HP (Postulate 1). Bisect HP (I, 10), then with the middle point of HP as a center and one-half HP as a radius describe a circle (Postulate 3). The circumference of this circle will obviously pass through the points H and P. It must also pass through N and K, otherwise the angles HNP and HKP would not be right angles (e). Again, bisect NK (I, 10) and with its middle point as a center and one-half NK as a radius describe another circle (Postulate 3). The circumference of this circle will also pass through the points N, K, P, and H for the same reason as the one above. Therefore these circumferences will coincide with one another (III, 10). Therefore there can be but one center point which being in both the lines NK and HP must be at the point of intersection O. Therefore the two triangles NOH and POK are equal to each other (I, 4), and therefore NH equals PK.



Q. E. D.

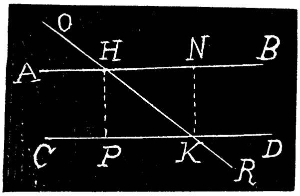
COROLLARY. The intercepts on two parallel straight lines by two common perpendiculars are equal to each other.

For, the triangles NOP and HOK are equal to each other (I, 4). Therefore NP is equal to HK, being homologous sides of two equal triangles.

(g)

PROPOSITION VII. *If a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another, etc.* (I, 29).

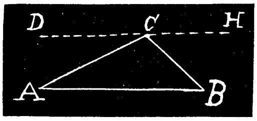
PROOF. Let the straight line OR fall on the two parallel straight lines AB and CD, meeting them in points H and K respectively. Then the angles BHK and CKH shall be equal to one another.



From H draw HP perpendicular to CD, and from K draw KN perpendicular to AB (I, 12). Then HP is also perpendicular to AB and KN is also perpendicular to CD (d). Therefore HP equals NK (f), and HN equals PK (f Corollary). Therefore the two triangles HPK and HNK are equal to each other (I, 8), and therefore the angle NHK equals the angle HKP, being homologous angles of two equal triangles. Q. E. D.

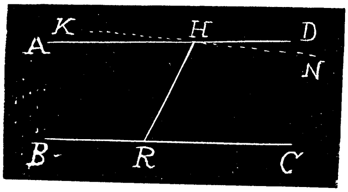
PROPOSITION VIII. *The sum of the angles of every plane triangle is equal to two right angles.*

PROOF. Let ABC be any plane triangle, then the sum of the angles A, B, and C is equal to two right angles. Through one of its vertices as C draw DH parallel to AB (I, 31). Then the angles A and DCA are equal to one another (g), as are also the angles B and HCB for the same reason. But the sum of the angles DCA, ACB, and BCH is equal to two right angles (I, 13). Therefore the sum of the angles A, B, and ACB must equal two right angles. Q. E. D.

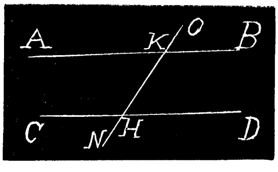


PROPOSITION IX. *Through a given point without a given straight line only one line can be drawn parallel to the given line.*

PROOF. Let BC be a straight line and H a point without. Draw AD through H parallel to BC (I, 31). Then no other line can be drawn through H parallel to BC. If possible suppose KN drawn through H parallel to BC. Then since the angles KHR and AHR are each equal to the angle HRC (g), they are equal to each other (Axiom 1), a part to the whole which is impossible. Therefore KN cannot be parallel to BC. Q. E. D.



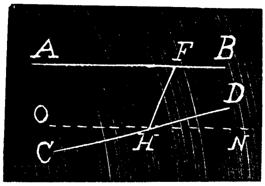
PROPOSITION X. *If a straight line fall on two parallel straight lines, the sum of the two interior angles on the same side of that line shall be equal to two right angles.*



PROOF. Let the straight line ON fall on the two parallel straight lines AB and CD. Then the sum of the two angles AKH and CHK is equal to two right angles. For, the sum of the two angles CHK and KHD is equal to two right angles (I, 13) and the angle AKH equals the angle KHD (g). Therefore, substituting the latter for the former we have the sum of the two angles AKH and CHK equal to two right angles. Q. E. D.

PROPOSITION XI. *If a straight line meet two straight lines so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.* Euclid, Axiom 12.

PROOF. Let the straight line FH meet the two straight lines AB and CD, making the two angles BFH and FHD together less than two right angles, then AB and CD shall meet, if continually produced, on that side of FH towards B and D. Since the angles BFH and AFH are together equal to two right angles, they must be greater than the sum of the two angles BFH and FHD. Therefore, the angle AFH must be greater than the angle FHD. Hence, draw the line ON through H making the angle FHN equal to the angle AFH (I, 23). Then ON is parallel to AB (I, 27). Therefore CD cannot be parallel to AB (i), and therefore CD and AB must meet if sufficiently produced. Since the sum of the angles AFH and FHO equals two right angles (j), the sum of the angles AFH and FHC must be greater than two right angles. Therefore AB and CD cannot meet on that side of FH toward A and C for then we should have a triangle the sum of whose angles would be greater than two right angles which is impossible by (h). Therefore they must meet on that side of FH toward B and D.



Q. E. D.

ZERO, INFINITESIMALS, INFINITY, AND THE FUNDAMENTAL SYMBOL OF INDETERMINATION.

By GEORGE LILLEY, Ph. D., Professor of Mathematics, State University, Washington.

The following is an outline of the method I use in explaining to the student in algebra how zero is used as a multiplier and a divisor, and how infinitesimals and infinity are used as divisors; also, interpretations of the results obtained by their use.

If we multiply a by a number that decreases by 1 each time beginning with any number, as $+4$, and continue the multiplication until -4 is reached, each product will decrease by a. Thus,

a	a	a	a	a	a	a	a
$+4$	$+3$	$+1$	*zero*	-1	-2	-3	-4
$+4a$	$+3a$	$+a$,	zero,	$-a$	$-2a$	$-3a$	$-4a$,

where zero is a constant number and obtained by subtracting *any* number from itself, as, $a-a=\textcircled{1}$, $\textcircled{1}$ representing *absolute* zero.

Evidently a multiplied by zero is one a less than a multiplied by $+1$, or

$a\times ⓛ=ⓛ$; also, a multiplied by -1 is one a less than a multiplied by zero, or $a\times -1=-a$. Similarly $a\times -2=-2a$, $a\times -3=-3a$, etc.

Hence, *If a constant number be multiplied by zero, the product is zero.*

Division may be defined as the process of finding how many times the divisor can be subtracted from the dividend and leave zero.

Dividing 12 by a number that decreases by unity each time, beginning with $+3$, we have

$$\frac{12}{+3}=4,\ \frac{12}{+2}=6,\ \frac{12}{+1}=12,\ \left(\frac{12}{\text{zero}}\right),\ \frac{12}{-1}=-12,\ \frac{12}{-2}=-6,\ \frac{12}{-3}=-4\text{; etc.}$$

The quotient 4 means that only 3 times $+4$ can be subtracted from 12 and leave zero ; and so on for the other quotients.

Since the divisor decreases by unity, the divisor one less than 1 is zero. The divisors less than zero are -1, -2, -3, etc., respectively. Then, the quotient, when zero becomes the divisor, must be between the quotients given by taking $+1$ and -1 as divisors, or between $+12$ and -12.

Then $\frac{12}{\text{zero}}$ or $\frac{12}{ⓛ}=\ominus$, where $\ominus$ represents no number of times. That is, there is *no number of times zero* that the divisor, zero, can be subtracted from 12 and *leave absolutely nothing.*

Since negative numbers are less than zero, ⓛ is not the least divisor of 12, or of any other number. If $\frac{12}{ⓛ}$ =infinity, or the *largest possible number*, 12 divided by -1 can not give -12 for a quotient. If $\frac{12}{-1}=-12$, $\frac{12}{\text{zero}}$ or $\frac{12}{ⓛ}$ can not give infinity for a quotient, for the divisor, -1, is one less than the divisor ⓛ

Hence, in general, $\frac{a}{ⓛ}=\ominus$.

For the quotient, $\ominus$, means that there is no number of times zero that the divisor, ⓛ, can be subtracted from a and leave zero.

Hence, *If a constant number be divided by zero, the quotient is no number of times.*

It is a consequence of confounding the 0, arising from dividing a by *infinity*, with the absolute zero, that so much confusion has been created in the discussions on this subject. All absolute zeros are constants. The other 0's, used in these discussions, are infinitesimals and variables, and may be less than ⓛ.

Since an infinitesimal can be subtracted from a an infinite number of times and leave zero ; therefore, $\frac{a}{◎}=\infty$, where ◎ represents an infinitesimal.

That is, *If a constant number be divided by an infinitesimal, the quotient is infinity.*

Suppose, for illustration, we divide a by a number that diminishes one-

tenth each time, beginning with one ; we will have the series

$$\frac{a}{1}=a,\ \frac{a}{\frac{1}{10}}=10a,\ \frac{a}{\frac{1}{100}}=100a,\ \frac{a}{\frac{1}{1000}}=1000a,\ \frac{a}{\frac{1}{10000}}=10000a,\ \ldots\ldots$$

Evidently, by continuing the series indefinitely, the divisor becomes *less* than any assignable number however small, and the value of the quotient increases without limit and becomes *greater* than any assignable number however great.

Hence, *If a constant number be divided by a decreasing variable, as the variable becomes too small to be expressed, the quotient becomes too large to be expressed.*

Since infinity can be subtracted from a the infinitesimal part of once and leave zero ; therefore, $\frac{a}{\infty}=\circledcirc$.

That is, *If a constant number be divided by infinity, the quotient is infinitesimal.*

Suppose the divisor, in the above illustration, increases each time, beginning with 1 ; we will have the series

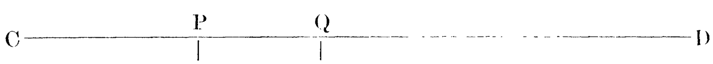
$$\frac{a}{1}=a,\ \frac{a}{10}=.1a,\ \frac{a}{100}=.01a,\ \frac{a}{1000}=.001a,\ \frac{a}{10000}=.0001a,\ \ldots\ldots$$

Evidently, by continuing the series indefinitely, the divisor becomes greater than any assignable number, however great, and the value of the quotient decreases without limit and becomes *less* than any assignable number, however small.

Hence, *If a constant number be divided by an increasing variable, as the variable becomes too great to be expressed, the quotient becomes too small to be expressed.*

This subject is also illustrated in interpreting the results obtained by assigning different values for the rates of travel and the distance apart, in Clairaut's problem, of the Couriers.

"Two Couriers, A and B, travel in the same direction, CD, at the rates m and n miles an hour, respectively. If at any time, say 12 o'clock, A is at P, and B is at Q, a miles from P, at what time and at what place are they together?"



Let t=the number of hours traveled, after 12 o'clock, to the place where A overtakes B, and d=the number of miles travelled by A in t hours ; or the number of miles from P to the place where A overtakes B.

Since the number of miles travelled by each, after 12 o'clock, equals the rate multiplied by the number of hours, we have

$$d=mt, \text{ and } d-a=nt.$$

Solving these equations, we have

$$t=\frac{a}{m-n}, \quad d=\frac{am}{m-n}.$$

We will now examine these values under different conditions.

1. If $m>n$.

This condition makes the values of t and d *positive*. That is, the Couriers are together *after* 12 o'clock, and at some place to the *right* of P.

If $m<n$.

This condition makes the values of t and d *negative*. That is, the Couriers are together *before* 12 o'clock, and to the *left* of P. This interpretation corresponds with the conditions made. For, if m is less than n, A travels more slowly than B, and it follows that they must have been together before 12 o'clock, and before they could have advanced as far as P.

3. If $m=n$, or $m-n=$zero.

Then $t=\frac{a}{0}$, and $d=\frac{am}{0}$.

As there is no number of times zero that subtracted from a leaves zero, there is no number of hours when they have been or will be together. Furthermore, as every number of times zero subtracted from a leaves a; that is, $a-v\times 0=a$, where v represents any number whatever, they are always the same distance apart.

Hence, *A result in the form* $\frac{a}{0}$ *indicates that the problem is impossible.*

This interpretation corresponds with the supposition made. For, if m is equal to n, the Couriers travel at the same rate, and since they were a miles apart at 12 o'clock, it is evident they never could have been, and never will be, together.

4. If $a=$zero, and $m>n$, or $m<n$.

Then $t=\frac{0}{m-n}$, and $d=\frac{0}{m-n}$.

They are together at the start, as shown by $a=$zero; but, as there is no number of times $m-n$ that subtracted from zero, will leave zero, they can never be together again.

Furthermore, the longer the time, the greater or less will $m-n$ be; hence, they will be constantly diverging. For example,

In 1 hour, zero$-(m-n)=n-m$, distance apart;
In 2 hours, zero$-2(m-n)=2n-2m$, distance apart;
In 3 hours, zero$-3(m-n)=3n-3m$, distance apart, etc.

Hence, $t=\frac{0}{m-n}$ indicates that they will be together in zero hours after 12 o'clock, but never after or before. For zero$-0\times(m-n)=$zero, is the only value for t that will satisfy the conditions.

Similarly, $d=\dfrac{\textcircled{1}}{m-n}$ means no distance from P, and shows that they were placed together by the conditions of the problem, $a=$zero, but for *all other time the problem is impossible.*

5. If $a=$zero, and $m=n$.

Then $t=\dfrac{\textcircled{1}}{\textcircled{1}}$, and $d=\dfrac{\textcircled{1}}{\textcircled{1}}$.

As any number of zeros subtracted from zero gives zero ; that is, zero$-v\times\textcircled{1}=$zero, where v represents any number whatever, they are together at all times ; for $t=$*any number.*

Hence, $t=\dfrac{\textcircled{1}}{\textcircled{1}}$ means all conceivable times, and $d=\dfrac{\textcircled{1}}{\textcircled{1}}$ means all conceivable distances, and are indeterminate, not being one, but every value.

Therefore, *A result* $\dfrac{\textcircled{1}}{\textcircled{1}}$ *indicates that the problem is indeterminate.*

The form, $\dfrac{\textcircled{1}}{\textcircled{1}}$, is a *symbol of indetermination*, and does not indicate that no solution can be found, but that too many can be determined. The indetermination consists in the fact that any one of the infinite solutions will answer just as well as any other.

January 11, 1897.

EDITORIALS.

This issue of the MONTHLY was delayed on account of securing sorts for Dr. Lovett's article.

Prof. E. L. Brown, formerly of the Capital University, Columbus, Ohio, is now a member of the Faculty of the Department of Mathematics of the Colorado State University.

The articles on "Euclidean Geometry Without Disputed Axioms," and "Zero, Infinitesimals, Infinity, and the Fundamental Symbol of Indetermination," are published at the request of the authors. They invite criticism on their respective articles, and, if there is any defect in the reasoning by which they arrived at their conclusions, they desire to have the same pointed out.

BOOKS AND PERIODICALS.

A Text-Book of Light. With Numerous Diagrams and Examples. By R. Wallace Stewart, D. Sc., London, Author of "An Elementary Text-Book of Heat and Light," "An Elementary Text-Book of Magnetism and Electricity," etc.

Third Edition. 8vo Cloth, 208 pages. Price, 3*s.* 6*d.* London : W. B. Cleve University Correspondence College Press. New York : Hinds & Noble, 4. Cooper Institute.

This little treatise on Light is clearly, neatly, and accurately written. The author as a teacher and writer needs no introduction. His works all bear evidence of a master of the subject under consideration. This book in the hands of the student will enable him to read with interest and profit the investigations in this most fascinating phenomenon of nature. B. F. F.

On the Transitive Substitution Groups that are Simply Isomorphic to the Symmetric or Alternating Group of Degree Six. By Dr. G. A. Miller.

This is a reprint of a paper read before the American Philosophical Society May 7, 1897, and published in the proceedings of that Society. B. F. F.

New Principles of Geometry with Complete Theory of Parallels. By Nicolái Ivánovish Lobachévski. Translated from the Russian by Dr. George Bruce Halsted. Volume fifth of the Neomonic Series.

The publication of the translation of this little pamphlet of 26 pages promised by Dr. Halsted at the Mathematical Congress of the World's Columbian Exposition was delayed for a personal visit to Kazan, the home of Lobachévski, and Maros Vásárhely, the home of Bolyai.

In this pamphlet is some interesting matter for the student of Non-Euclidean Geometry, and it should be read by every teacher of geometry. Several of our readers have written to us saying that they could see no sense in the Non-Euclidean Geometry. We say to these, read this little pamphlet, then with the light you get from it turn back and read the first and all subsequent articles of Dr. Halsted's which have appeared in the MONTHLY during the last four years, then turn to other sources for information on the subject. After having done this, you will find that there is a Non-Euclidean Geometry, that its argument is as rigorous as the Euclidean, and that its deductions are equally interesting. B. F. F.

A Text-Book of Physics. Largely Experimental, including the Harvard College "Descriptive List of Elementary Exercises in Physics" By Edwin H. Hall, Ph. D., Professor of Physics in Harvard College, and Joseph Y. Bergen, A. M., Instructor in the Harvard Summer School of Physics, and Junior Master in English High School, Boston. Revised and Enlarged. 8vo Cloth. 596 pages. New York : Henry Holt & Co.

This book needs no introduction to the public. The first edition has proved its usefulness in all experimental courses in Physics. The second edition is even an improvement over the first. B. F. F.

Ordinary Differential Equations. An Elementary Text-Book with an Introduction to Lie's Theory of the Group of One Parameter. By James Morris Page, Ph. D., University of Leipzig, Fellow by Courtesy Johns Hopkins University, Adjunct Professor of Pure Mathematics in University of Virginia. 8vo Cloth. 226 pages. New York and London : The Macmillan Co.

This is the best elementary exposition of the subject of Ordinary Differential Equations with which we are acquainted. It differs from the older text-books upon the subject in one important respect, namely, in the method of treatment. Instead of giving theories of integration for certain classes of Differential Equations, as for instance, the Homogen-

eous or Linear Differential Equations as is done in the older works on the subject, the author has followed the method of Professor Lie. In 1870, Lie showed that it is possible to subordinate all the older methods or theories of integration to a general method. By the method of Lie it is possible to derive all of the older theories from a common source and at the same time build a broader foundation for the general theory of Differential Equations. The simple, elegant, and clear presentation of the subject in this work makes it possible for a student who has ambition and a fair knowledge of Analytical Geometry and Calculus to master this book without an instructor. B. F. F.

On the Primitive Substitution Groups of Degree Fifteen. By Dr. G. A. Miller. Pamphlet, 12 pages.

This paper is an extract from the Proceedings of the London Mathematical Society, Vol. XXVIII., and is along Dr. Miller's favorite line of investigation. B. F. F.

Higher Arithmetic. By Wooster Woodruff Beman, Professor of Mathematics in the University of Michigan, and David Eugene Smith, Professor of Mathematics in the Michigan State Normal School. 12mo. Cloth and Leather Back. 194 pages. Price, 80 cents. Chicago: Ginn & Co.

Among the many valuable features of this work are the elimination of the traditional problems which have become the common property of nearly every arithmetic published during the last quarter century; the introduction, instead of the traditional problems, of simple problems arising in the study of elementary physics, as, for example, easy problems in Electrical Measurements, problems coming under the application of Boyle's Law, the law of Falling Bodies, Specific Gravity, etc.; the treatment of the Metric System in the first part of the book (page 59) and the frequent use made of it in the subsequent part; the introduction of the common graphic methods of representing statistics; and the complete omission of rules. The entire omission of rules is a very common feature of the arithmetics which are published at the present time and of those which have been published during the last three or four years. It is my belief that all rules that can not be established easily by the deductive method of reasoning should be set in *good print* in the arithmetics. This is especially the case with the rules in Mensuration. The student of arithmetic is in general not competent to follow the argument which establishes the rule for finding the area of a triangle when three sides are given. Yet it is better for the student that he commit this rule to memory, though he does not know how it is established, than to be ignorant of its existence and the means by which the area of triangles are computed when the sides are given. It seems to me that any arithmetic treating the subject of mensuration ought to give the rules for finding the surface and volume of the three round bodies, the area of parallelograms, circles, and triangles, the triangles having the base and altitude given or the three sides. To these might be added the rules for finding the surface and volume of prisms and pyramids. Aside from these I heartily believe in the omission of rules. The work before us does not omit the consideration of the most of the above geometrical magnitudes, but the rules are not expressly stated. This work of Professors Beman and Smith is, however, one that we can very cheerfully recommend. B. F. F.

A Brief Introduction to the Infinitesimal Calculus. Designed Especially to Aid in Reading Mathematical Economics and Statics. By Irving Fisher, Ph. D., Assistant Professor of Political Science in Yale University, Co-author of Phillips and Fisher's Elements of Geometry. 12mo. Cloth. 84 pages. Price, 75 cents. New York and London: The Macmillan Co.

This little work on the Calculus will be received with joy by a great army of stud-

ents, teachers, and professors, who have lacked the time and courage to attack some of the more exhaustive works on the subject yet felt the need of a knowledge of the Calculus in order to enable them to read with intelligence the highest authorities on economic as well as other subjects. Dr. Fisher has prepared this little work with a special view of the needs of this class of students. Any one with a clear mind can very easily read and understand every sentence in this book. There is no metaphysical speculation nor obscure statements made in establishing its first principles.

In considering the formula $s=\frac{1}{2}gt^2$, where s=space a body falls under the influence of gravity in the time t, he says, pages 2 and 3, "Since the above formula holds true of *all* points, it holds true now, when the time is $t+\triangle t$ and the distance $s+\triangle s$. That is $s+\triangle s=16(t+\triangle t)^2$. This gives $s+\triangle s=16t^2+32t.\triangle t+16\triangle t^2$. But $s=16t^2$. Subtracting, we have

$$\triangle s=32t.\triangle t+16(\triangle t)^2,$$

$$\text{whence } \frac{\triangle s}{\triangle t}=32t+16\triangle t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

This is the *average* velocity during the small interval $\triangle t$.

Thus, if $\triangle t=\frac{1}{2}$ second and t be five seconds, the average speed of the body during that half second (viz., the one beginning 5 seconds from the rest) is $32\times 5+19\times\frac{1}{2}$, or 168 per second. If we take $\frac{1}{160}$ of a second instead of $\frac{1}{2}$, we have $32\times 5+16\times\frac{1}{160}$, or 168.1 feet per second.

The speed at the very instant of completing the 5th second is obtained by putting $\triangle t=0$, which gives 160 as the instantaneous speed.

Now when $\triangle t=0$, we call it dt, because 0 would not remind us of the kind of quantity which vanished, whereas dt does suggest t, the magnitude which vanished. When $\triangle t$ becomes 0, or dt, $\triangle s$ evidently becomes zero too, for a body can not go any distance in no time. This zero we call ds. Equation (1) therefore becomes at the limit

$$\frac{ds}{dt}=32t+[16]dt,$$

$$\text{or } \frac{0}{0}=32t+0, \text{ which may be written}$$

$$\frac{ds}{dt}=32t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

for we can neglect the zeros on the right, but not those on the left (the ratio of two zeros does not reduce to zero).

* * * * * * * * * * * * *

It may be objected to the reasoning in the last article that $\frac{ds}{dt}$ or $\frac{0}{0}$ is indeterminate. This is true. $\frac{0}{0}$ is equal to 2, or 19, or 1, or any number we we please. *But the limit* $\frac{\triangle s}{\triangle t}$ *is not indeterminate.* We thus use $\frac{ds}{dt}$ in two distinct senses, viz: $\lim\frac{\triangle s}{\triangle t}$ and $\frac{\lim\triangle s}{\lim\triangle t}$.

The first is determinate, the second is indeterminate, though for that very reason it may always be put equal to the first. Only the first, or $\lim\frac{\triangle s}{\triangle t}$ is important. This is the ultimate ratio of two vanishing quantities."

Excepting the statement in the next to the last sentence, viz., that $\lim \frac{\triangle s}{\triangle t}$ is determinate and $\frac{\lim \triangle s}{\lim \triangle t}$ indeterminate, we claim that Dr. Fisher has established the fundamental principles of the Differential Calculus in a simple, rigorous, and logical manner. By the principle of Limits, the $\lim \frac{\triangle s}{\triangle t} = \frac{\lim \triangle s}{\lim \triangle t}$, that is to say, the limit of the quotient of two variables equals the quotient of their limits. Then if one is determinate the other is determinate, or if one is indeterminate the other is indeterminate. They are, however, both determinate. The statement that dt is used instead of 0 to preserve the trace of the quantity that vanished will be considered by many mathematicians as the rankest sort of mathematical heresy, the reanimating of Berkeley's "ghost of departed quantities." But here too Dr. Fisher's position is absolutely impregnable, for, since $\frac{0}{0}$ is, *per se*, indeterminate, but determinate by the equation by which, in every case, it is defined, it may be replaced by the ratio of any two quantities which preserves the ratio that defines $\frac{0}{0}$. So in the case above, $\frac{0}{0}$ can be replaced by $\frac{ds}{dt}$ or $\frac{y}{x}$ or the quotient of any other two quantities which preserve the ratio $32t$. But $\frac{0}{0}$ is replaced by $\frac{ds}{dt}$ to preserve the trace of the quantities which vanished, and ds and dt can represent large or small parts of s and t. There are in general three possible ways by which the ratio $32t$ can be preserved by $\frac{ds}{dt}$. First, ds being considered a constant, and dt a variable; second, ds being a variable, and dt a constant; third, $\frac{ds}{dt}$ both being variables. Each of these three ways of viewing $\frac{ds}{dt}$ is used in the Calculus. This method of exposition is used in my classes with the result that students are enabled to use the Calculus, not as a machine by which to grind out problems, but as an instrument of research.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, \$2.50 per year in advance. Single Number, 25 cents. The American Monthly Review of Reviews Co., 13 Astor Place, New York City.

The American Monthly Review of Reviews for October has several articles of unusual interest to women readers. Miss Frances Willard tells the story of the World's W. C. T. U. movement; Mrs. Ellen M. Henrotin, president of the General Federation of Women's Clubs, outlines the benefits of those organizations; Mrs. Sheldon Amos, of England, writes of a London Women's Club, and Miss Mary Taylor Blauvelt contributes an enlightening article on the opportunities for women at the English universities. B. F. F.

The Arena. An Illustrated Monthly Magazine. Edited by John Clark Ridpath, LL. D. Price, \$2.50 per year in advance. Single Number, 25 cents. Boston: The Arena Co.

The Arena for October continues the battle for reform. The number is especially interesting and in some parts brilliant; it is in all parts aggressive and courageous. Hon. Charles A. Towne's article, "The New Ostracism," is in the author's best vein of critical analysis. In the course of the discussion he attacks with great vigor the plutocratic interference with professors in colleges and universities. Herman E. Taubeneck continues his cogent statistical attack on consecrated wealth. Judge Walter Clark sends out a powerful plea for the establishment of public rights over semi-public interests and institutions. The Editor of *The Arena* continues with unabated vigor his onslaught on the organized forces of plutocracy. His article, "Prosperity: the Sham and the Reality," is one of his strongest and best. Dr. Ridpath's exposition of the bottom purposes and methods of the money power is as caustic as it is true. Mary Platt Parmelee's article on the Political Philosophy of the Father of American Democracy is an original and forceful argument for popular liberties. B. O. Flower is again at his best pace in "The Latest Social Vision," in which he discusses the merits of Bellamy's "Equality." Perhaps the most radical and defiant article in the number is "The Dead Hand in the Church," by Rev. Clarence Lathbury, in which he attacks with destructive criticism the domination of the dead past over the living present in the church. "Hypnotism in its Scientific and Forensic Aspects" is the subject of an interesting and useful article by Marion L. Dawson. "Suicide: Is It Worth While ?" is the caption of Charles B. Newcomb's startling study of one of the most interesting and painful themes of the age. The "Plaza of the Poets" is rich with the contributions of Ironquill, Junius Hempstead, Clinton Scollard, Reubie Carpenter, and Helena M. Richardson; while "The Editor's Evening" sparkles with its usual gems of social and poetical philosophy. Under "Book Reviews" the charming poems of Madison Cawein are set forth with merited commendation.
B. F. F.

The Open Court. A Monthly Magazine Devoted to the Science of Religion, and the Religion of Science, and the Extension of the Religious Parliament Idea. Edited by Dr. Paul Carus. T. J. McCormack, Assistant Editor, and E. C. Hegeler and Mary Carus, Associate Editors. Price, $1.00 per year in advance. Single Copies, 10 cents. Chicago and London: The Open Court Publishing Co.

The following is the table of contents of the November number: "An Introduction to the Study of Ethnological Jurisprudence," by the Late Justice Albert Hermann Post, Bremen, Germany; "History of the People of Israel from the Beginning to the Destruction of Jerusalem by the Chaldeans," by C. H. Cornill, Professor of Theology in the University of Koenigsberg; "The Religion of Science; the Worship of Beneficence," by James Odgers Knutsford, England; "Death in Religious Art," by the Editor; "Vivisection from an Ethical Point of View: A Controversy," by Prof. Henry C. Mercer, and others; "Leonhard Euler," a biographical sketch by T. J. McCormack; "The Sacred Books of the Buddhists," by Albert J. Edmunds; "Brief Notes on some Recent French Philosophical Works;" Book Reviews, and Notes. Among the book reviews is a just estimate or criticism of "Finkel's Mathematical Solution Book;" the review contains about a page and a half, and is written by Assistant Editor T. J. McCormack.
B. F. F.

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. IV. NOVEMBER, 1897. No. 11.

BIOGRAPHY.

VASILIEV.

BY GEORGE BRUCE HALSTED.

ALEXANDER VASILIEVITCH VASILIEV was born August 24 (old style), 1853, at Kazan. His father, orientalist already academician, was then Professor of Chinese Literature at the University of Kazan. His mother was a daughter of Simonov, Professor of Astronomy in Lobachévski's time and his predecessor as Rector. In 1855 on the transference of the Oriental Faculty to St. Petersburg, Vasiliev's father removed thither. In 1870 Vasiliev finished the course of the fifth St. Petersburg gymnasium as gold-medalist.

The love for mathematics, awakened in the gymnasium, where in Class VI. he studied Sturm's Differential Calculus, carried him to the mathematical department of the University of St. Petersburg, which then boasted Somov and the great Chebishev (Tchébychev).

As result of his earnest studies for 1870-73 appears the work "On the separation of roots," crowned with a gold medal. In 1874 on his taking his first degree he was invited by the University of Kazan to begin there his teaching as Privat-docent. Though he had planned to continue his studies at Berlin, he accepts this invitation to his birthplace and begins in November, 1874.

His Dissertatio pro venia legendi was entitled "On the separation of the roots of simultaneous equations." In January 1875 he begins to lecture on Functiontheory, all his scholars being older than the professor.

ALEXANDER VASILIEVITCH VASILIEV.

His thesis for the Master's examination, taken in 1878, was "On singular solutions in connection with the new views on the problem of integration of differential equations of first order."

His Master's Dissertation, accepted in May 1880, he prepared abroad, spending the year 1879 in Berlin with Weierstrass and Kronecker, and in Paris with Hermite. His subject was "On the rational functions analogous to the double-periodic." Soon after he was made Docent in the University of Kazan. He spent the next summer in Germany, and wrote "The teaching of mathematics in Berlin and Leipzig Universities."

A question which had so long interested him was treated again in his Doctor's dissertation in 1884, "Theory of the separation of the roots of systems of simultaneous equations." Now chosen Professor Extraordinarius, he was made Professor Ordinarius in 1887.

In 1884 Vasiliev was made president of the physico-mathematical section of the Scientific Society of Kazan University. In 1891 this section changed itself into the independent "Physico-mathematic Society." The eight volumes of Proceedings of this section from 1880 to 1890 contain a series of important articles and criticisms by Vasiliev. Since 1883 he has been the authority on all Russian works in Analysis for the "Fortschritte der Mathematik." In the years 1880-89 Vasiliev was particularly active as member of the local assembly, the Zemstvo, in the government of Kazan. By his influence, the number of folk-schools increased in 1883-89 from 43 to 90, of scholars from 1692 to 3100. Thus his district, Svijaschsk, attained a first rank in all Russia by passing from one scholar for 920 inhabitants to one scholar for 28 inhabitants.

Since 1891 Vasiliev has edited the "Bulletin de la Société Physico-Mathématique de Kasan," which now exchanges with 110 learned publications. In the brilliantly successful celebration of the hundredth birthday of Lobachevski by this society, and the foundation of the Lobachevski Prizes, more than a thousand persons from all over the world took part as subscribers.

The position now held by Vasiliev in the Russian mathematical world may be judged from his being chosen by the Academy of Sciences to report on a great work offered in competition for the Buniakovski Prize. The book received the half prize, while Vasiliev's report is to be honored by insertion in the Transactions of the Academy and the award of the Buniakovski Medal.

The great International Congress of Mathematicians just born into permanent life at its wonderfully successful first meeting, in Zurich, and next to meet at Paris, owes its inception to Vasiliev, who pushed the idea into prominence in every country. It was on his initiative that I brought the matter up in the American Mathematical Society and obtained the signatures of all the members present at the Brooklyn meeting to an endorsement of the idea giving specific credit to Vasiliev as originator. At the actual congress he was most active. From him, Laisant, and G. Cantor emanated the three important resolutions constituting the three commissions of the Congress.

The many works of Vasiliev, being inaccessible because in Russian, will

not be enumerated, but the depth of his thinking and charm of his style may be judged from his great Address on Lobachévski, which it was my good fortune to give to the world in a *literal* translation, not a paraphrase. This translation was greeted by a tremendous outburst of enthusiasm in the mathematical world.

It must here suffice to give a few detached sentences from a mass of letters sent me. "I am astonished to find these researches of such deep philosophical import," writes Professor Daniels of the University of Vermont. "I have read it with intense interest," says Cajori. "This life and work of Lobachevski will be a grand inspiration to mathematicians," says Zerr. "I rejoice that you, 'in the midst of the virgin forests of Texas,' are able to do this work," says Professor Carman. "It will arouse a deeper enthusiasm for scientific achievement and widen the horizon of every reader. Surely no mathematician should miss this gem from farthest Russia," says Dr. L. E. Dickson. "By translating this most interesting Address, you have earned for yourself a title to the thanks of the mathematical world," says Dr. Paul Staeckel, since so well known in this very line. I sent this translation in 1894 to Professor Friedrich Engel of Leipzig, to whom I afterward offered for translation into German my translation of Lobachevski's largest work, "New principles of Geometry with complete theory of parallels." He issued the Address in 1895, saying in his *Nachwort:* "Ich habe die Wassiljefsche Rede nach dem Original uebersetzt, obwohl bereits eine englische Uebersetzung von *G. B. Halsted* (Austin, Texas, 1894) vorlag ; es schien mir aber fuer einen Deutschen nicht passend, eine russische Schrift nach einer englishen Uebersetzung zu uebertragen. Selbstverständlich habe ich aber die Halstedsche Uebersetzung ueberall verglichen und bekenne gern, dass sie mir an manchen Stellen gute Dienste geleistet hat."

A French translation and an (incomplete) Spanish translation have since appeared.

This transcendently beautiful production, linking forever the name of Vasiliev with that of Lobachevski, wins both for author and object, the love of every reader.

A personal picture with scene at Kazan the ancient capital of the Tartars, must be reserved for a subsequent chapter : "A Visit to Vasiliev."

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from October Number.]

LVIII. Fig. 29.

$ALMI$ is equivalent to $2IAC-2BAE$ is equivalent to $ACDE$.

$BKML$ is equivalent to $BKNC$ is equivalent to $BCFH$.

$\therefore$ $ABKI$ is equivalent to $ACDE+BCFH$.

LIX. Fig. 29.

$QIK = RAB$. $BOP = AFQ$. $OHKP = DEAR$.

$\therefore$ $ABKI$ is equivalent to $ACDE + BCFH$.

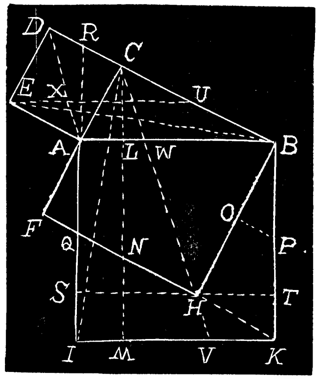


Fig. 29.

LX. Fig. 29.

BHK is equivalent to $AFQ + DEAR$.

Then $BAIK$ is equivalent to $BRQK$.

$\therefore$ $ABKI$ is equivalent to $ACDE + ACFH$.

LXI. Fig. 29.

$ABTS$ is equivalent to $2ABH$ is equivalent to $BCFH$.

$STKI = ALMI$ is equivalent to $ACDE$.

$\therefore$ $ABKI$ is equivalent to $ACDE + BCFH$.

LXII. Fig. 29.

Same as in LXI, except that

$STKI$ is equivalent to $ABUE$ is equivalent to $ACDE$.

LXIII. Fig. 29.

$WBKV$, the half of $ABKI$, is equivalent to $BWH + BHK + HVK$.

But $BHK = BCA$ is equivalent to $BWC + DXE$; and $HVK = AXE$.

$\therefore$ $\frac{1}{2}ABKI$ is equivalent to $\frac{1}{2}ACDE + \frac{1}{2}BCFH$.

$\therefore$ $ABKI$ is equivalent to $ACDE + BCFH$.

LXIV. Fig. 30.

$MAF = NFA$. Then, $KLI = FCD$.

$ILN = DEM$. $BHK = BCA$.

$\therefore$ $ABKI$ is equivalent to $ACDE \pm BCFH$.

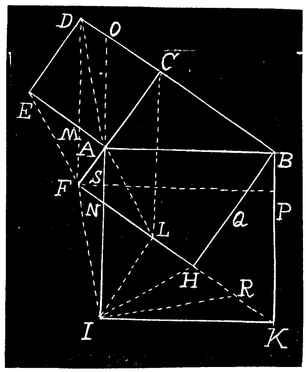


Fig. 30.

LXV. Fig. 30.

$KHI = DEF$ is equivalent to $\frac{1}{2}ACDE$.

$HIL = ALF$.

$ILA = DEF$ is equivalent to $\frac{1}{2}ACDE$.

$BHK = BCA$.

$\therefore$ $ABKI$ is equivalent to $ACDE + BCFH$.

LXVI. Fig. 30.

$LNOC$ is equivalent to $LFDC - NFDO$ is equivalent to $ACDE$.

For $LFDC$ is equivalent to $ACDE + 2FAE$, and $2FAE$ is equivalent to $2FAD$ is equivalent to $NFDO$.

Also, $KLCB$ is equivalent to $BCFH$.

$\therefore$ $KNOB$ is equivalent to $ACDE + BCFH$.

But, $ABKI$ is equivalent to $KNOB$.

$\therefore$ $ABKI$ is equivalent to $ACDE + BCFH$.

LXVII. Fig. 30.

$ISPK$ is equivalent to $2IFK=2ADB$ is equivalent to $2ACB+ACDE$ is equivalent to $ACB+FHQ+ACDE$.

$SABP$ is equivalent to $FABQ$.

$\therefore ABKI$ is equivalent to $ACDE+BCFH$.

LXVIII. Fig. 30.

$ILR=ACD$, and $ILF=AED$.

Then $IRK=IFA$. $BHK=BCA$.

$\therefore ABKI$ is equivalent to $ACDE+BCFH$.

LXIX. Fig. 30.

$LNOC$ is equivalent to $2LAC=2FED$ is equivalent to $ACDE$.

$KLCB$ is equivalent to $BCFH$.

$\therefore ABKI$ is equivalent to $KNOB$ is equivalent to $ACDE+BCFH$.

[To be Continued.]

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M. (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from October Number.]

PROPOSITION XXIX. *Resuming Fig. 33 of the preceding proposition: I say every straight AC, which cuts angle BAX, finally at a finite or terminated distance (even in hypothesis of acute angle) will meet BX in a certain point P, if only AC be produced ever more toward the parts of the points X.*

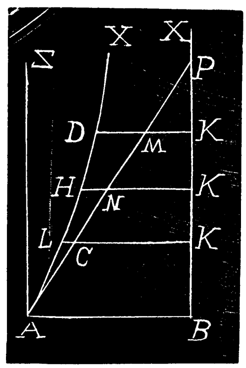


Fig. 33.

Proof. And firstly indeed (lest straight AC include space with AX) it must meet at finite distance the straights LK, HK, DK in certain points C, N, M; must meet, I say, unless before (and that at a finite distance, just as we maintain) it meets BX in some point between the point B and one of the points K.

Then (from Corollary I. after XXIII.) the angles ACK, ANK, AMK will be obtuse.

Moreover those angles, always obtuse, approach (from the preceding proposition) without any certain limit, to equality with a right angle, when indeed that AC is supposed to meet BX only at an infinite distance. Therefore such an ordinate KMD can be reached that at it the angle AMK

exceeds a right angle by less than the angle DAC. But then angle DAC, or DAM, together with angle AMD will be greater than a right angle. Wherefore, the obtuse angle ADM being added, the three angles together of the triangle ADM will be greater than two right angles, which is against the hypothesis of acute angle.

Therefore every straight AC, which cuts that angle BAX, finally at a finite or terminated distance (hypothesis of acute angle) must meet BX in a certain point P. Quod etc.

COROLLARY I. Hence no straight AZ, which toward the parts of the points X makes an acute angle greater than BAX can ever meet BX, either at a finite, or at an infinite distance. For as far as so should happen, now AX dividing angle BAZ, ought (against the premised supposition) to meet BX at a finite distance, as this is demonstrated of the straight AC dividing angle BAX.

COROLLARY II. Moreover it follows that no determinate acute angle will be the maximum of all under which a straight line produced from point A meets BX at finite distance. For if toward the parts of the point X you assume any point higher than the point P, it follows that the straight joining point A with this higher point will make with AB a greater angle than angle BAP. And so ever without any intrinsic end. Wherefore angle BAX (since indeed AX both always approaches to BX, and meets it only at an infinite distance) will be the outside limit of all acute angles under which straights produced from that point A meet the aforesaid BX at a finite distance.

[To be Continued.]

SOPHUS LIE'S TRANSFORMATION GROUPS.

A SERIES OF ELEMENTARY, EXPOSITORY ARTICLES.

By EDGAR ODELL LOVETT, Princeton University.

II.

THE GROUP OF ONE PARAMETER. THE INFINITESIMAL TRANSFORMATION. EXISTENCE OF AN INFINITESIMAL TRANSFORMATION IN A GROUP OF ONE PARAMETER.

5. Consider the plane as a point manifoldness, *i. e*, a space whose space element is the point. The plane will then be two-dimensional,* *i. e* contain ∞^2

*This idea and its bearing in the paragraph are emphasized here not to introduce any unnecessary ultra refinement but because of their use in geometrical illustrations to appear in succeeding articles. For example, the plane is one, two, three, or four dimensional according as, a circle with fixed center, the straight line, a circle of general position, or parabola, be taken as space element, since there are ∞^1, ∞^2, ∞^3, ∞^4, of these elements respectively in the plane. Similarly if the straight line is element it has no dimension, the point has one dimension, the plane two dimensions, and ordinary space four dimensions.

elements, or in other words the position of a point in the plane will be determined by two parameters, the coördinates of the point.

A point-transformation of the plane into itself is an operation by which every point in the plane is conveyed into the position of some point in the same plane. In order to represent this operation analytically, let us take as the coördinate system of reference an ordinary rectangular Cartesian system, x, y ; then the point transformation is expressed by two equations of the form

$$x_1=\varphi(x, y), \quad y_1=\psi(x, y), \tag{1}$$

where (x, y) is the original point and (x_1, y_1) the transformed point. It is further assumed that the transformation is of such a nature that every point (x_1, y_1) of the plane may be regarded as having originated from some point in the plane by effecting the transformation. This geometrical assumptions finds its analytical condition in the demand that the two functions $\varphi(x, y)$ and $\psi(x, y)$ be independent functions and thus the preceding equations are soluble theoretically with regard to x and y. For, suppose that φ and ψ were not independent, and for example let

$$\varphi=n\alpha(x, y), \quad \psi=m\alpha(x, y) ;$$

then eliminating $\alpha(x, y)$ from the equations of the transformation

$$x_1=n\alpha(x, y), \quad y_1=m\alpha(x, y)$$

we find that the points (x, y) of the plane are transformed into the points (x_1, y_1) of the straight line

$$y_1=\frac{m}{n}x_1,$$

and hence point of general position no longer is conveyed into point of general position by the transformation.

If the equations (1) be solved with regard to the variables x, y, there result two equations of the form

$$x=\bar{\varphi}(x_1, y_1), \quad y=\bar{\psi}(x_1, y) \tag{2}$$

which represent a transformation that carries the point (x_1, y_1) back into the position (x, y) ; this transformation (2) is called the *inverse* of the transformation (1). If the transformation (1) be followed by the transformation (2), that is, if the two transformations be carried out successively, we have the two equations

$$x_1=x, \quad y_1=y.$$

These equations are very particular cases of equations (1) and hence should represent a transformation. The transformation which they represent obviously transforms a point into itself, or in other words, it leaves all points at rest, for this reason it is called the *identical* transformation.

6. If the equations of a transformation

$$x_1 = \varphi(x, y, a), \quad y_1 = \psi(x, y, a) \tag{3}$$

contain an arbitrary constant a, these equations no longer represent a single transformation but a family of ∞^1 transformations, since the arbitrary parameter a may assume all values from $-\infty$ to $+\infty$. Let us make the hypothesis that the equations (3) represent such a family of transformations that *the successive application of any two transformations of the family is equivalent to a transformation belonging to the same family*; in this case the family (3) is called a *group*; since the group contains *one* parameter a and hence ∞^1 transformations, the group is called a *one parameter group*, or a *group* of one parameter, or symbolically a G_1. Further, since the parameter varies continuously the group is said to be a *continuous* group. As the group contains a finite number of parameters, in this case but one, namely a, it is a *finite continuous group*. The sentence above in italics expresses the group property of the family. A footnote in the preceding article calls attention to the fact that the group property is peculiar to certain classes of families and not common to all of them.

The analytical criterion for a one-parameter group as just defined reveals itself in the following manner. The transformation

$$T_1 \qquad x_1 = \varphi(x, y, a), \quad y_1 = \psi(x, y, a), \tag{4}$$

changes the point (x, y) into the point (x_1, y_1); let T_1 be followed by the transformation T_2 which corresponds to the value a_1 of the parameter and changes the point (x_1, y_1) into the point (x_2, y_2), given by the equations

$$T_2 \qquad x_2 = \varphi(x_1, y_1, a_1), \quad y_2 = \psi(x_1, y_1, a_1). \tag{5}$$

The transformation, T_3, say, which will carry the point (x, y) directly into the position (x_2, y_2) is found by eliminating (x_1, y_1) from the equations (4) and (5). The elimination yields

$$T_3 \equiv T_1 T_2 \begin{cases} x_2 = \varphi\{\varphi(x, y, a), \psi(x, y, a), a_1\}, \\ y_2 = \psi\{\varphi(x, y, a), \psi(x, y, a), a_1\}. \end{cases} \tag{6}$$

If this transformation is to belong to the original family it must be capable of expression in the form

$$x_2 = \varphi(x, y, \lambda), \quad y_2 = \psi(x, y, \lambda),$$

where λ is a certain function of a and a_1 alone.

Hence the criterion sought is that the two equations

$$\varphi\{\varphi(x, y, a), \psi(x, y, a), a_1\} \equiv \varphi\{x, y, \lambda(a, a_1)\},$$

$$\psi\{\varphi(x, y, a), \psi(x, y, a), a_1\}\equiv\psi\{x, y, \lambda(a, a_1)\},$$

must exist identically for all values of x, y, a, and a_1.

7. In the sequel we shall study only those continuous groups* which contain the *inverse* transformation of every transformation of the group. *i. e.* to a tranformation corresponding to the parameter a,

$$T_1 \qquad x_1=\varphi(x, y, a), \quad y_1=\psi(x, y, a),$$

there corresponds a transformation of the family whose parameter is $\bar{a}$ say,

$$T_2 \qquad x_2=\varphi(x_1, y_1, \bar{a}), \quad y_2=\psi(x_1, y_1, \bar{a}),$$

such that T_2 cancels T_1 and gives

$$x_2=x, \quad y_2=y,$$

the identical transformation.

Accordingly, if the transformations of a group are inverse in pairs the group contains the *identical* transformation. Let a_0 be the value of the parameter which gives the identical transformation, then

$$\varphi(x, y, a_0)\equiv x, \quad \psi(x, y, a_0)\equiv y.$$

A transformation of the family whose parameter is $a_0+\delta a$, where δa is an indefinitely small quantity will move the point (x, y) through only an infinitesimal distance, such a transformation is called an *infinitesimal transformation*, where by an *infinitesimal* transformation of the group is meant a transformation whose parameter differs by an infinitesimal from that value of the parameter which gives the *identical transformation.*

8. There is a most intimate connection between the notions *infinitesimal transformation* and *one parameter group.* It is proposed to derive now three fundamental theorems of LIE which establish this relationship. The first proves that *every one parameter group contains an infinitesimal transformation*, the second that *every infinitesimal transformation generates a one parameter group*, and the third that *a one parameter group contains but one infinitesimal transformation.*

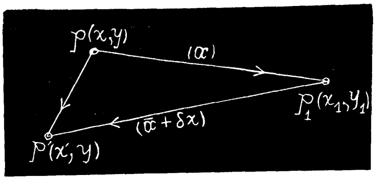
The three theorems show that an infinitesimal transformation may be taken as the representative of a one parameter group.

That a group of one parameter contains an infinitesimal transformation may be seen geometrically in the following manner :

Let a transformation of the group which corresponds to the parameter α and which is designated for convenience by (α) carry the point p (x, y) to the position p_1, (x_1, y_1). By assumption the inverse of (α) is contained in the group. Let the parameter of this inverse transformation be $\bar{\alpha}$; $\bar{\alpha}$ is a certain

*The reader must be reminded that this limitation is really not a restriction. LIE has proved in volume III of the Theory of Transformation Groups, Theorem 26, that the defining equations of any continuous group can be derived from those of a group whose transformations are inverse in pairs.

function of α. The transformation $(\bar{\alpha})$ changes all points p_1 into the points p again respectively. A transformation whose parameter differs infinitesimally from $\bar{\alpha}$, say $\bar{\alpha}+\delta\alpha$, will carry the point p_1 not back to p, but to a position at an infinitesimal distance from p, say p'. The successive performance of (α) and $(\bar{\alpha}+\delta\alpha)$ will carry p to p_1 and then to p'; but (α) and $(\bar{\alpha}+\delta\alpha)$ belong to the group, hence the third transformation to which they are equivalent belongs to the group ; that is, the transformation which carries p to p', a point infinitesimally near, belongs to the group, or in other words the group contains an infinitesimal transformation.



This geometric process may now be clothed in analytic garb. The first transformation (α) is given by the equations

$$x_1=\varphi(x,\ y,\ \alpha),\quad y_1=\psi(x,\ y,\ \alpha)\ ; \tag{7}$$

the second transformation $(\bar{\alpha}+\delta\alpha)$ by

$$x'=\varphi(x_1,\ y_1,\ \bar{\alpha}+\delta\alpha),\quad y'=\psi(x_1,\ y_1,\ \bar{\alpha}+\delta\alpha). \tag{8}$$

The elimination of x_1, y_1 from these equations gives the transformation which carries p to p', namely

$$x'=\varphi\{\varphi(x,\ y,\ \alpha),\ \psi(x,\ y,\ \alpha),\ \bar{\alpha}+\delta\alpha\},\ y'=\psi\{\varphi(x,\ y,\ \alpha),\ \psi(x,\ y,\ \alpha),\ \bar{\alpha}+\delta\alpha\}. \tag{9}$$

Developing* these values in powers of $\delta\alpha$ we have

$$\begin{aligned}x'&=\varphi\{\varphi(x,\ y,\ \alpha),\ \psi(x,\ y,\ \alpha),\ \bar{\alpha}\}+\frac{\partial\varphi\{\varphi(x,\ y,\ \alpha),\ \psi(x,\ y,\ \alpha),\ \bar{\alpha}\}}{\partial\bar{\alpha}}\,\frac{\delta\alpha}{1}+\ldots\ ,\\ y'&=\psi\{\varphi(x,y,\ \alpha),\ \psi(x,\ y,\ \alpha),\ \bar{\alpha}\}+\frac{\partial\psi\{\varphi(x,\ y,\ \alpha),\ \psi(x,\ y,\ \alpha),\ \bar{\alpha}\}}{\partial\bar{\alpha}}\,\frac{\delta\alpha}{1}+\ldots\end{aligned} \tag{10}$$

Now since the transformetions (α) and $(\bar{\alpha})$ are inverse

$$\varphi\{\varphi(x,\ y,\ \alpha),\ \psi(x,\ y,\ \alpha),\ \bar{\alpha}\}\equiv x,\quad \psi\{\varphi(x,\ y,\ \alpha),\ \psi(x,\ y,\ \alpha),\ \bar{\alpha}\}\equiv y\ ;$$

hence the equations of the transformation which changes p into p' are

$$\begin{aligned}x'&=x+\frac{\partial\varphi\{\varphi(x,\ y,\ \alpha),\ \psi(x,\ y,\ \alpha)\ \bar{\alpha}\}}{\partial\bar{\alpha}}\,\frac{\delta\alpha}{1}+\ldots\ ,\\ y'&=y+\frac{\partial\psi\{\varphi(x,\ y,\ \alpha),\ \psi(x,\ y,\ \alpha),\ \bar{\alpha}\}}{\partial\bar{\alpha}}\,\frac{\delta\alpha}{1}+\ldots\ ,\end{aligned} \tag{11}$$

and in this form they represent an infinitesimal transformation since the values

*It is to be remarked once for all that all functions here considered are regular analytic functions and hence expansible by Taylor's Theorem.

of x' and y' differ from x and y respectively by infinitely small quantities. It is easy to see that the coefficients of $\delta\alpha$ do not vanish, for if we put for $\varphi(x, y, \alpha)$ and $\psi(x, y, \alpha)$ their equals x_1 and y_1 respectively, these coefficients equated to zero are

$$\frac{\partial\varphi(x_1, y_1, \bar{a})}{\partial\bar{a}}\equiv0, \quad \frac{\partial\psi(x_1, y_1, \bar{a})}{\partial\bar{a}}\equiv0.$$

But these last identities assert that φ and ψ are free from $\bar{a}$, that is, in general the equations of the group contain no parameter which is contrary to hypothesis.

The quantity $\bar{a}$ is a function of α, since to a transformation $(\bar{a})$ there corresponds, by hypothesis, a completely determinate inverse transformation $(\bar{a})$. The equations (1) of the infinitesimal transformation may be written in the form

$$x'=x+\xi(x, y, \alpha)\delta\alpha+\ldots\ldots, \quad y'=y+\eta(x, y, \alpha)\delta\alpha+\ldots\ldots*$$

LIE thus arrives at the following theorem :

I. *Every one parameter group whose transformations are inverse in pairs contains at least one infinitesimal transformation.*

Princeton University, 22 October, 1897.

[To be Continued.]

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

83. Proposed by the late REV. G. W. BATES, A. M., Pastor of M. E. Church, Dresden City, Ohio.

A has three notes; the first and second, \$1000 each, and the third \$457; all dated April 1, 1884. The first is due April 1, 1888, second, April 1, 1889, and the third, April 1, 1890, and each bearing interest at 6%. What must B pay for the three notes September 21, 1886, that the investment will bring him 8% compound interest ?

Solution by G. B. M. ZERR, A. M., Ph. D., President of Russell College, Lebanon, Mo.

(I). Regarding the notes as bearing simple interest. We get

\$1000 × 1.24=\$1240, amount of first note.

\$1000 × 1.30=\$1300, amount of second note.

\$457 × 1.36=\$621.52, amount of third note.

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*These equations contain a constant a which can be arbitrarily chosen, hence we can find an infinitesimal transformation of the group in many different ways. But the sequel will show that all these, excepting a constant factor, are identical in their terms of the first order of infinitesimals.

From September 21, 1886, to April 1, 1888, is $1\frac{19}{36}$ years.
From September 21, 1886, to April 1, 1889, is $2\frac{19}{36}$ years.
From September 21, 1886, to April 1, 1890, is $3\frac{19}{36}$ years.
Let x=amount paid for first note ; y, for second ; z, for third.
$\therefore x(1.08)^{1\frac{19}{36}}=1240$, or $\log x=\log 1240-1\frac{19}{36}\log 1.08$.
$\therefore x=\$1102.448$.
$y(1.08)^{2\frac{19}{36}}=1300$, or $\log y=\log 1300-2\frac{19}{36}\log 1.08$.
$\therefore y=\$1070.176$.
$\log z=\log 621.52-3\frac{19}{36}\log 1.08$.
$\therefore z=\$473.743$.
$x+y+z=\$2646.367$=whole amount to be paid for the notes.
(II). If the notes bear compound interest we get,
$\$1000\times(1.06)^4=\$1262.477$, amount of first note.
$\$1000\times(1.06)^5=\$1338.226$, amount of second note.
$\$457\times(1.06)^6=\$648.263$, amount of third note.
$\therefore \log x=\log 1262.477-1\frac{19}{36}\log 1.08$.
$\therefore x=\$1122.43$.
$\log y=\log 1338.226-2\frac{19}{36}\log 1.08$.
$\therefore y=1101.646$.
$\log z=\log 648.263-3\frac{19}{36}\log 1.08$.
$\therefore z=\$494.127$.
$x+y+z=\$2718.20$=whole amount paid for the three notes.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

78. Proposed by J. A. MOORE, Ph. D., Professor of Mathematics, Millsaps College, Jackson, Miss.

Required the number of normals that can be drawn from any point (a, b) to the parabola $y^2=2px$.

I. Solution by the PROPOSER.

The equation of the normal to the parabola in terms of its slope, (s), is

$$y=sx-\tfrac{1}{2}(sp)(2+s^2)\ldots\ldots\ldots\ldots(1).$$

Substituting a, b for x, y in (1), and putting the equation in a new form, we have,

$$s^3+\tfrac{1}{2}p(p-a)s+(2b/p)=0\ldots\ldots\ldots\ldots(2).$$

Denoting Sturm's functions by F, F_1, F_2, etc., we have the following :

$F=s^3+(2/p)(p-a)s+(2b/p)$.

$F_1=3s^2+(2/p)(p-a)$.

$F_2=-2(p-a)s-3b$.

$F_3=-b^2-(8/27p)(p-a)^3$.

Consider five cases.

(1). Suppose $p-a<0$, and $(8/27p)(p-a)^3$ numerically greater than b^2. Sturm's Theorem gives

	F	F_1	F_2	F_3
For $s=+\infty$,	$+$	$+$	$+$	$+$
$s=-\infty$,	$-$	$+$	$-$	$+$

Hence the roots are real and unequal.

(2). Suppose $p-a<0$, and $(8/27p)(p-a)^3$ numerically less than b^2. Then

	F	F_1	F_2	F_3
$s=+\infty$,	$+$	$+$	$+$	$-$
$s=-\infty$,	$-$	$+$	$-$	$-$

Hence, there is one real root.

(3). Suppose $p-a>0$. Then

	F	F_1	F_2	F_3
$s=+\infty$,	$+$	$+$	$-$	$-$
$s=-\infty$,	$-$	$+$	$+$	$-$

Hence, one real root.

(4). Suppose $-b^2-(8/27p)(p-a)^3=0$.

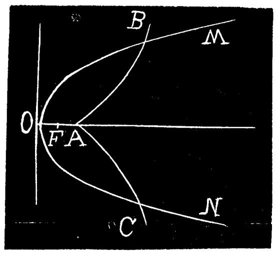
Then there are equal roots, as in this case F and F_1 have a common divisor, and all the roots are real.

(5). Suppose $-b^2-(8/27p)(p-a)^3=0$, and $p=a$.

Then $b=0$, and all the roots are equal, each being 0.

Hence if MON is the given parabola and BAC its evolute, that is, the semi cubical parabola whose equation is

$$b^2=\frac{8}{27p}(a-p)^3.$$



Then, (1), if the point (a, b) is within (to the right) of the evolute, three normals can be drawn to the parabola ; (2), if the point (a, b) is on the evolute, but not at A, two normals can be drawn ; (3), if the point (a, b) is A, or is without (to the left) of the evolute, one normal can be drawn to the parabola.

II. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio, and OTTO CLAYTON, Fowler, Ind.

If m be the tangent of the angle which the normal makes with the axis of x, the normal is given by

$$y=mx-pm-(\tfrac{1}{2}p)m^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

This passing through (a, b) gives

$$b=am-pm-(\tfrac{1}{2}p)m^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

a cubic in m, showing that the required number is three.

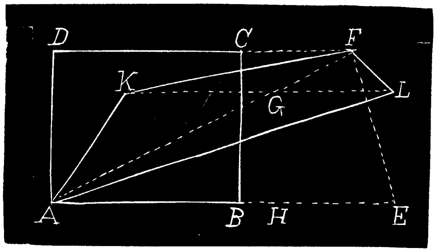
Also solved by *G. B. M. ZERR* and *J. F. LAWRENCE.*

79. Proposed by JOHN MACNIE, Professor of Mathematics, University of North Dakota, University, N. D.

To construct a quadrilateral of given area, the diagonals, one of which is given, cutting each other in given ratios and at a given angle.

I. Solution by JAS. F. LAWRENCE, Freshman Class, Classfcal Course. Drury College, Springfield, Mo., and the PROPOSER.

Let AC be a rectangle equivalent to the given area and having a side AB equal to one-half of the given diagonal. Produce AB to E, so that $BE=AB$; at A construct $\angle EAF$ equal to the angle to be made by the diagonals, and let AF meet DC produced in F. Divide AF in G in the ratio of division of one diagonal, and AE in H, in the ratio of the given diagonal. On an indefinite line drawn through G parallel to AB lay off GK, CL, equal to AH, HE, respectively, and FK, FL, AK, AL; $AKFL$ is the required quadrilateral.



Join EF. $\triangle FAE \simeq AC$. It is also equivalent to $AKFL$; for each is equivalent to one-half the parallelogram formed by drawing parallels through the extremities of the diagonals AF, LK.* Hence $AKFL \simeq AC$; it has also a diagonal $KL=AE=2AB$; its diagonals also are divided in the given ratios, and make an angle $FGL=\angle EAF=$ the given angle. Hence, etc.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Lebanon, Va.; OTTO CLAYTON, Fowler, Ind.; and F. R. HONEY, Ph. B., New Haven, Conn.

Let AB be the given diagonal, COB the given angle, $\triangle$ the given area, $m:n$ the given ratio for the known diagonal, $p:q$ the given ratio for the unknown diagonal.

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*From the well known theorem: Any quadrangle is equivalent to one-half the parallelogram formed by drawing lines through its vertices parallel to its diagonals; follow the corollaries—

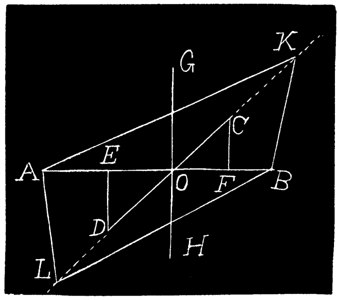
a. Two quadrangles are equivalent if their diagonals are respectively equal and intersect at the same angle. (Triangle a special case.)

b. Any quadrangle is equivalent to the rectangle of its diagonals multiplied by half the sine of their angle.

Let x=unknown diagonal, $\beta=\angle COB$, h=altitude of triangle above AB, h_1=the altitude of the triangle below AB, a=given diagonal.

$$x=2\triangle/a\sin\beta \quad \text{.........................} (1).$$

Divide AB at O in ratio $m:n$ and draw the indefinite line KL making an angle β with AB.



Let $CO=p$, $DO=q$, $OK=y$, $OL=z$.

$\therefore\ y=xp/(p+q)=2p\triangle/\{a(p+q)\sin\beta\}$,

$z=xq/(p+q)=2q\triangle/\{a(p+q)\sin\beta\}$.

$\therefore\ h=2p\triangle/\{a(p+q)\}$

$$h_1=2q\triangle/\{a(p+q)\}.$$

Draw OG, OH perpendicular to AB and $=h$, h_1; draw GK, LH parallel to AB, cutting KL in K and L. $\therefore$ $AKBL$ is the quadrilateral.

III. Solution by A. H. BELL, Hillsboro, Ill., and F. R. HONEY, Ph. B., New Haven, Conn.

Let a, and b, equal the segments of the given diagonals.

Let $x+y$, and $x-y$, equal the segments of the other diagonals.

Let r=the given ratio of the later diagonal, and θ the angle. Then

$$\frac{x+y}{x-y}=r \quad \text{..........................} (1).$$

Also, $ax\sin\theta+bx\sin\theta=c$ the given area.................................(2).

$$\text{(2) and (1) } x=\frac{c}{(a+b)\sin\theta}=\frac{r+1}{r-1}y \quad \text{................} (3).$$

$$y=\frac{c(r-1)}{(a+b)(r+1)\sin\theta},\quad x+y=\frac{2cr}{(a+b)(r+1)\sin\theta},\quad x-y=\frac{2c}{(a+b)(r+1)\sin\theta}.$$

Now having all the segments of the two diagonals with the given angle between them. the construction of the required quadrilateral is very simple.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

52. Proposed by S. ELMER SLOCUM, Union College, Schenectady. New York.

A chain 16 feet long is hung over a smooth pin with one end 2 feet higher than the other end and then let go. Show that the chain will run off the pin in about 7-5 second. [*Wright's Mechanics*, page 92.]

I. Solution by **WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.**

Let 16 feet$=2a$, 9 feet$=b$, and $x=$the length of the longer part of chain at any time t from the beginning of motion. Then, if m be the mass of a unit of length of the chain, $g=32$, the equation of motion is

$$\frac{d}{dt}\left(2ma\frac{dx}{dt}\right)=2mg(x-a) \quad \ldots\ldots\ \ldots\ldots\ldots\ldots(1).$$

Multiplying both members by $\frac{d(x-a)}{dt}$ and integrating,

$$a\left(\frac{d(x-a)}{dt}\right)^2=g(x-a)^2+C \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

When $x=b$, $\frac{d(x-a)}{dt}=0$, and $C=-g(b-a)^2$;

$$\therefore\ (2) \text{ is } \sqrt{a}\frac{d(x-a)}{dt}=\sqrt{g}\sqrt{(x-a)^2-(b-a)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$\text{or } dt=\sqrt{\frac{a}{g}}\cdot\frac{d(x-a)}{\sqrt{(x-a)^2-(b-a)^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Integrating, $t=\sqrt{\frac{a}{g}}\log\left(x-a+\sqrt{(x-a)^2-(b-a)^2}\right)+C'$ ......(5).

When $x=b$, $t=0$; $\therefore$ $C'=-\sqrt{\frac{a}{g}}\log(b-a)$, and (5) then becomes

$$t=\sqrt{\frac{a}{g}}\log\left\{\frac{x-a+\sqrt{(x-a)^2-(b-a)^2}}{b-a}\right\} \quad \ldots\ldots\ldots(6).$$

Introducing numbers, $t=1.38$ seconds.

II. Solution by **ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Miss., and HENRY HEATON, M. Sc., Atlantic, Iowa.**

Let s denote the distance through which the lower end of the string descends in t seconds.

Then, since the acceleration equals the moving force divided by the mass moved,

$$\frac{d^2s}{dt^2}=\frac{2+2s}{16}g.$$

Integrating, $\left(\frac{ds}{dt}\right)^2=\frac{1}{8}g(s^2+2s)$, no constant being added since when

$s=0$, $\frac{ds}{dt}=0$. From the last equation $\frac{ds}{\sqrt{s^2+2s}}=\frac{\sqrt{2g}}{4}dt$ which, by integration, gives $\log_e(s+1+\sqrt{2s+s^2})=\frac{\sqrt{2g}}{4}t$, the constant again being zero, since when $t=0$, $s=0$, and $\log 1=0$.

Taking this between the limits 7 and 0, $t=\frac{7}{5}$, approximately.

Also solved by *G. B. M. ZERR*, *C. W. M. BLACK*, *J. SCHEFFER*, and the *PROPOSER*.

53. **Proposed by J. C. NAGLE, M. A., C. E., Professor of Civil Engineering, Agricultural and Mechanical College of Texas.**

Find the locus of the center of gravity of an arc of constant length for a parabola.

Solution by G. B. M. ZERR, A. M., Ph. D., President of Russell College, Lebanon, Va.

Let u, v be the coördinates of the center of gravity, $y^2=4ax$, be the equation to the parabola for any point on the curve.

$$\therefore su=\int_0^x x\,ds=\tfrac14(a+2x)\sqrt{ax+x^2}-\frac{a^2}{8}\log\left(\frac{a+2x+2\sqrt{ax+x^2}}{a}\right)$$

$$=\tfrac14(a+2x)\sqrt{ax+x^2}-\frac{a^2}{4}\log\left(\frac{\sqrt{x}+\sqrt{a+x}}{\sqrt{a}}\right)$$

$$\frac{a^2}{4}\log\left(\frac{\sqrt{x}+\sqrt{a+x}}{\sqrt{a}}\right)=\frac{a}{4}(s-\sqrt{ax+x^2}).$$

$$\therefore su=\tfrac12\sqrt{x}(a+x)^{\frac32}-\tfrac14as \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

$$sv=\int_0^x y\,ds=\tfrac43\sqrt{a}(a+x)^{\frac32}-\tfrac43a^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

a and x are both variable in (1) and (2). It does not appear easy to eliminate a and x and thus obtain an equation in u, v.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

53. **Proposed by A. H. BELL, Hillsboro, Illinois.**

Given $x^2-114\tfrac14y^2=\mp3$ to find the least values of x and y in integers.

I. Solution by the PROPOSER.

It can be demonstrated that D, in $x^2-Ay=\pm D$, can be any denominator of the complete quotients from the $\sqrt{A}$, and that x and y are the numerator and denominator of the convergent preceding the term in which D is taken. Now the complete quotients for the $\sqrt{114\frac{1}{4}}$ are

$$\frac{0+\sqrt{114\frac{1}{4}}}{1};\quad \frac{10\frac{1}{2}+\sqrt{114\frac{1}{4}}}{4};\quad \frac{9\frac{1}{2}+\sqrt{114\frac{1}{4}}}{6};\quad \frac{8\frac{1}{2}+\sqrt{114\frac{1}{4}}}{7};\ \text{etc.}$$

No. term	.....	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12,	etc., reversing.
$\therefore \sqrt{114\frac{1}{4}}=\frac{1}{0}$, $\frac{10\frac{1}{2}}{1}$	:	5,	3,	2,	1,	2,	1,	6,	2,	1,	1,	10,	10,	etc., reversing.
Complete denom'rs$=1$	:	4,	6,	7,	12,	6,	14,	3,	8,	9,	12,	2,	2,	etc., reversing.

Hence x and y are found in the 6th convergent and also the 15th convergent, and x and y=2095 and 196, and also from the 15th term x=42, 307, 834, and y=3, 958, 154. [Also see problem 38.]

II. Solution by JOSIAH H. DRUMMOND, LL. D., Portland Maine.

I have not solved this problem as stated, but as I have solved it in this form $x^2-114\frac{1}{4}y^2\pm3=\square$ ......(1), and as that is a pretty question, I send my solution.

Multiplying by 4, it becomes $4x^2-457y^2\pm12=\square$=say $(2x-m^2=4x^2-4mx+m^2)$, from which we find

$$x=\frac{457y^2+m^2\pm12}{4m}.$$

$(m\pm12)/4m$ evidently becomes integral when $m=6$; and we have

$$x=\frac{457y^2}{24}+2,\ \text{or}\ \frac{457y^2}{24}+1.$$

$457y^2/24$ becomes integral when $y=12n$, and $x=2742n+2$, or $=2743n+1$, according as the $+$ or $-$ sign before 3 is taken.

If $n=1$, $y=12$, and x=2744 or 2743; in the former, 3 is negative, and in the latter, positive, in order to make the expression a square.

54. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

In the expression $2x^2-2ax+b^2$, find two series of values for x in integral terms of a and b.

I. Solution by the PROPOSER.

$2x^2-2ax+b^2$ is evidently a square when $x=a$. Take $x=y+a$, and substituting, we have $2y^2+2ay+b^2=\square=(\text{say})(my-b)^2$.

Reducing, $y=2(a+bm)/(m^2-2)$. Taking $m=2/1$, 10/7, 58/41, etc., we have *one* integral series of the value of y, viz: $a+2b$, $49a+10b$, etc. Taking m=3/2, 17/12, 99/70, etc., we have *another* integral series of the value of y, viz:

$8a+12b$, $288a+408b$, etc. By adding a to each term of each series we have two series of the value of x. These series hold good when either a or b is zero ; but if both are zero, $x=0$.

It will be noticed that this solution applies the terms of the question to the expression $2x^2+2ax+b=\square$, the value of x in the latter being a less than in the former.

II. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

$2x^2-2ax+b^2=0$. $\therefore x=\frac{1}{2}(a\pm\sqrt{a^2-2b^2})$.

Let $a=p^2+2q^2$, $b=2pq$. Then $x=p^2$ or $2q^2$.

$\therefore p$	q	a	b	x,
2	1	6	4	4 or 2,
3	2	17	12	9 or 8,
4	3	34	24	16 or 18,
1	2	9	4	1 or 8,
etc.	etc.	etc.	etc.	etc.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

54. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

A man is at the center of a circle whose diameter is equal to three of his steps. If each step is taken in a perfectly random direction, what is the probability, (1), that he will step outside the circle at the second step, and, (2), that he will step outside at the third step ?

I. Solution by the PROPOSER.

Let O be the center of the circle, A, the end of the first step, and B, the end of the second, and C, the end of the third.

Let $\angle OAB=\theta$, $\angle OBC=\phi$, $OB=x$, and $OC=y$.

Then if the length of the step be taken as the unit of measure, $x=2\sin\frac{1}{2}\theta$, and $y=(x^2+1-2x\cos\phi)^{\frac{1}{2}}=(4\sin^2\frac{1}{2}\theta+1-4\sin\frac{1}{2}\theta\cos\phi)^{\frac{1}{2}}$.

If $x=\frac{3}{2}$, B falls upon the circumference of the circle, and $\theta=2\sin^{-1}\frac{3}{4}$. If θ be $>2\sin^{-1}\frac{3}{4}$, and $<\pi$, the second step falls outside the circle. The probability of this is $P_1=(\pi-2\sin^{-1}\frac{3}{4})/\pi$.

If θ be $<2\sin^{-1}\frac{3}{4}$, and $y=\frac{3}{2}$, C falls upon the circumference of the circle, and $4\sin^2\frac{1}{2}\theta+1-4\sin\frac{1}{2}\theta\cos\phi=\frac{9}{4}$ or $\phi_1=\cos^{-1}(\sin\frac{1}{2}\theta-5/16\sin\frac{1}{2}\theta)$. Hence if ϕ be $>\phi_1$ the third step falls outside the circle. The chance that ϕ will be $>\phi_1$ and $<\pi$ is $(\pi-\phi_1)\pi$. The chance that θ has any particular value is $d\theta/\pi$. Hence the probability that the third step falls outside the circle is

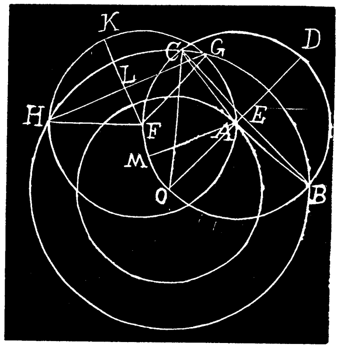
$$P_2=\frac{1}{\pi^2}\int_0^{2\sin^{-1}\frac{3}{4}}(\pi-\phi_1)d\theta.$$

This is not integrable in general terms but its value may be readily approximated by methods of mechanical quadrature.

II. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va., and J. A. MOORE, Ph. D., Professor of Mathematics, Millsaps College, Jackson, Miss.

Let $AO=a$, then $CO=\frac{3}{2}a$.

(1). Let his first step place him at the point A, then in order that he may step outside on the second step he must step somewhere on the arc CDB.



Let $AE=t$, $EC=u$, $\angle DAC=\beta$, P=chance in (1), p=chance in (2).

Now $u^2=a^2-t^2=\frac{9}{4}a^2-(a+t)^2$, $\therefore\ t=\frac{1}{8}a$.

$\therefore\ \cos\beta=\frac{1}{8}$.

$\therefore\ P=\beta/\pi=\cos^{-1}\frac{1}{8}/\pi=.460106$.

(2). Let chord $OM=\frac{1}{2}a$, then in order that he may step out the third step he must step somewhere on the arc CM or its equal on the opposite side

$$\angle CAM=\delta=\pi-(\beta+OAM)$$

$$=\cos^{-1}\left(\frac{3\sqrt{105}-7}{64}\right).$$

P_1=chance he steps on this arc$=\delta/\pi=.379034$.

If his second step places him on arc CM then his third step must place him on the arc GKH. The $\angle KFH$ may vary from 0 to $\cos^{-1}(-\frac{1}{3})$.

$\therefore\ p_1$=chance that he steps on arc $GKH=\dfrac{\cos^{-1}(-\frac{1}{3})}{2\pi}$

$\therefore\ p_1=.304086$.

Now $p=P_1\times p_1=\delta/\pi\times\dfrac{\cos^{-1}(-\frac{1}{3})}{2\pi}=.115259$.

Solved with a different result by *CHAS. C. CROSS*.

55. Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

It has been clear for 15 consecutive days, what is the chance of the 16th day being cloudy ?

Solution by the PROPOSER.

Let p=chance, p_1=chance that 16th day is clear.

$$\therefore\ p_1=\frac{\int_0^1 x^{16}dx}{\int_0^1 x^{15}dx}=\tfrac{16}{17}.\quad \therefore\ p=1-p_1=\tfrac{1}{17}.$$

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

49. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Give a general proof that the centre of gravity, or centroid, determines that point from which the sum of the distances to all other points of a given area is the minimum.

This problem is almost the same as No. 30, Miscellaneous, solutions of which were published on pages 334-5 of Vol. II, and on pages 86-88 of Vol. III. No further solutions have been received. If any of our contributors will attempt other solutions, they will be given in a future number. EDITOR.

50. Proposed by JOHN KEELEY ELLWOOD, A. M., Principal of the Colfax School, Pittsburg, Pa.

Describe and compute the actual path traversed by the moon in July and August, 1896, taking into account the motion of the earth around the sun.

No solution of this problem has been received. Dr. S. Hart Wright remarks that "a solution is not possible, as the *actual* path of the moon in space is required, while the moon and the earth describe, in their orbits, neither circles nor ellipses, but curved lines that are *undulatory*, being affected by perturbations due to other planets. If the orbits of the earth and moon were circles or ellipses, the moon's path would be an epicycloidal curve, always concave towards the sun." With the aid of a Nautical Almanac or data of the moon's path during the time asked, it would seem that a practically correct solution of the problem could be effected. We shall be pleased to publish anything further from contributors on this problem. EDITOR.

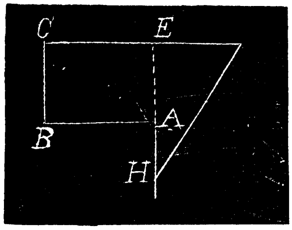
51. Proposed by F. M. SHIELDS, Coopwood, Miss.

A stock dealer traveled from his home H, due north across a lake L 40 miles wide to a city, and bought 156 horses and 177 mules for \$23631; he then traveled farther due north to A, and bought at same price 468 horses and 235 mules for \$52245; he then traveled from A due west 130 miles to B, and bought 120 cows; he then traveled due north to C, and bought 250 sheep; he then traveled from C due east 330 miles to D, and bought 300 goats,—paying 1-4 as much for cows as horses, and 1-9 as much for sheep as mules, and 1-2 as much for goats as sheep; at D he turned and traveled in a straight line to the city, a distance equal to the sum of the entire distance he traveled due north from his home H; he sold all his stock at a profit of 20%. How far did he travel from his home H the entire trip around and back to the city? What was the cost of each head of stock, and what was the entire gain?

I. Solution by P. S. BERG, A. M., Principal of Schools, Larimore, N. D.; CHARLES C. CROSS, Laytonsville, Md.; H. C. WILKES, Skull Run, W. Va.; J. SCHEFFER, A. M., Hagerstown, Md.; and G. B. M. ZERR, A. M., Ph. D., The Russell College, Lebanon, Va.

Let x=price of each horse, y=price of each mule.

Then $156x+177y=23631$; and $468x+235y=52245$.



$\therefore x=\$80$, $y=\$63$.

$\frac{1}{4}$ of $\$80=\$20$, price of each cow ; $\frac{1}{9}$ of $\$63=\$7$, price of each sheep ; $\frac{1}{2}$ of $\$7=\$3.50$, price of each goat. $120\times 20=2400$; $250\times 7=1750$; $300\times 3.50=1050$.

$\$2400+\$1750+\$1050+\$23631+\$52245=\$81076$, entire cost. 20% of $\$81076=\$16215.20$, entire gain.

Let $AH=u$, $BC=v$.

$\therefore (40+u+v)^2=(u+v)^2+(200)^2$.

$\therefore u+v=480$ miles.

$\therefore 480+40+480+40+330+130=1500$ miles.

II. **Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.**

Draw a diagram of the traveling, and produce line HA to E in line CD. Represent the city by O. Then OED be a right triangle in which $ED=330-130=200$ miles.

Put a=distance from home to city. Let $x=OE$; then $OD=x+a$. Whence $(x+a)^2=x^2+200^2$.

$$\therefore x=\frac{200^2-a^2}{2a},\ =\frac{200^2}{2a}-\tfrac{1}{2}a.$$

Now, in order that x may be positive, $\frac{1}{2}a<200^2/2a$; whence $a<200$.

But as the lake is 40 miles wide, a can not be less than 40. Therefore for positive values of x, a may have any value from 40 to 200.

The distance due north$=\dfrac{200^2-a^2}{2a}+a=\dfrac{200^2+a^2}{2a}$; and the entire distance traveled$=\dfrac{200^2+a^2}{a}+460$.

When $a=40$, or if H and O are situated on the lake, the entire distance traveled$=1500$ miles.

When $a=200$, $x=0$, and the city is the farthest north traveled. A would then coincide with O, and C with B.

When $a>200$, x is *negative*. Instead of traveling north from the city, he would then go west from the city to B, and thence *south*, the value of x, to C. For *any* positive value of x, A may be at any point in a due north line between O and E.

Let h, m, c, s, and g be the cost per head, respectively, of horses, mules, cows, sheep, and goats. Then $156h+177m=\$23631$, (1) ; $468h+235m=\$52245$, (2) ; $c=\frac{1}{4}h$, (3) ; $s=\frac{1}{9}m$, (4) ; and $g=\frac{1}{2}s$, (5). From (1) and (2), $h=\$80$, and $m=\$63$. Whence $c=\$20$, $s=\$7$, and $g=\$3\frac{1}{2}$.

$\therefore$ The stock cost $\$23631+\$52245+\$2400+\$1750+\$1050=\$81076$.

By selling his stock at a gain of 20%, he gained $\frac{1}{5}$ of $\$81076=\$16215.20$.

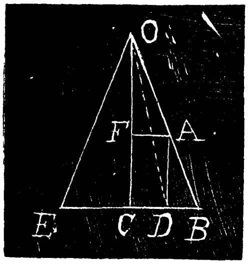
Also solved by *E. W. MORRELL*, and *JOSIAH H. DRUMMOND, LL. D.*

52. Proposed by I. J. WIREBACK, M. D., St. Petersburg, Penn.

What is the volume of a segment of a right cone, whose diameter is 6 inches and perpendicular 9 inches? The section being parallel with the perpendicular of the cone and includes $\frac14$ of its circumference at the base.

I. Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Mass.

Let OBE be section of cone perpendicular to section cutting off segment ADB. By considering projection of hyperbolic section AD on parallel plane through the axis, it is seen that the asymptotes are intersections of latter plane with conical surface. Accordingly, if $OF=a$, $FA=b$, equation to hyperbola is



$$a^2y^2-b^2x^2=-a^2b^2.$$

Now $FA=CD=\frac12$ side of square inscribed in circle $EB=\frac32\sqrt2=b$. $OF:FA=OC:CB$, or $OF:\frac32\sqrt2=9:3$; $OF=\frac92\sqrt2=a$. Substituting in formula for area of hyperbola,

$$\text{area } AD=(b/a)x\sqrt{x^2-a^2}-ab\log_e\left(\frac{x+\sqrt{x^2-a^2}}{a}\right),$$

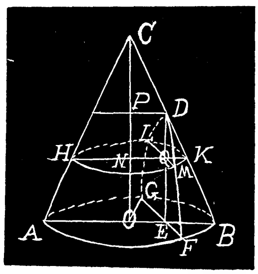
$$=\tfrac13\times9\sqrt{81-\tfrac{81}{2}}-27\log_e\left(\frac{9+\sqrt{81-\frac{81}{2}}}{\frac92\sqrt2}\right),$$

$$=\tfrac{27}{2}\sqrt2-\tfrac{27}{2}\log_e(\sqrt2+1).$$

Volume of conical segment $OAD=\frac13CD\times$area AD, $=\frac{27}{2}-\frac{27}{4}\sqrt2\log_e(\sqrt2+1)$.
Area of circular segment $BD=9\pi/4-\frac92$, $=\frac94(\pi-2)$.
Volume of conical segment $OBD=\frac13OC\times$area DB, $=\frac{27}{4}(\pi-2)$.
Volume $ABD=$volume $OBD-$volume $OAD=\frac{27}{4}(\pi-2)-\frac{27}{2}+\frac{27}{4}\sqrt2\log_e(\sqrt2+1)$
$=\frac{27}{4}\pi-27+\frac{27}{4}\sqrt2\log_e(\sqrt2+1)$, $=2.619+$ cubic inches.

II. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

Let $AFB-C$ be the cone, $GLDMF$ the section made by the plane cutting off the given segment. Let $AB=6=2R$, $OC=9=h$, $OE=c$. Since $\angle GOF=\pi/2$, $GF=R\sqrt2$.



$\therefore OE=\sqrt{OG^2-GE^2}=\sqrt{R^2-\frac12R^2}=\frac12R\sqrt2$
$=\frac12(3\sqrt2)=c$.

Let $CN=x$, then $CO:OB=CN:NK$,
or $h:R=x:NK$. $\therefore NK=Rx/h=NL$.

Similarly $CP=CO-DF=h-\dfrac{h(R-c)}{R}=\dfrac{hc}{R}$.

Area of segment $LMK=\dfrac{R^2x^2}{h^2}\cos^{-1}\left(\dfrac{ch}{Rx}\right)-\dfrac{c}{h}\sqrt{R^2x^2-h^2c^2}$.

$$\therefore\ V=\int_{ch/R}^{h}\left\{\frac{R^2x^2}{h^2}\cos^{-1}\left(\frac{ch}{Rx}\right)-\frac{c}{h}\sqrt{R^2x^2-h^2c^2}\right\}dx,$$

$$=\tfrac{1}{3}h\left\{R^2\cos^{-1}\left(\frac{c}{R}\right)-2c\sqrt{R^3-c^2}+\frac{c^3}{R}\log\left(\frac{R+\sqrt{R^2-c^2}}{c}\right)\right\},$$

$$=3\{9\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)-9+\frac{9\sqrt{2}}{4}\log(\sqrt{2}+1)\},\ =2.619 \text{ cubic inches.}$$

III. **Solution by J. SCHEFFER, A. M., Hagerstown, Md.**

I Solution. Designating the radius of the base by r, the altitude by h, and choosing the center of the base for the origin of orthogonal coördinates, CO for the axis of z, the radius OB for the axis of x and a radius parallel to the section FDG for that of y, we find the equation of the cone to be

$$z=(h/r)(r-\sqrt{x^2+y^2},$$

and the volume V of $COFGD$

$$=\int_{-\sqrt{r^2-x^2}}^{-\sqrt{r^2+x^2}}zdy=\frac{h}{r}\int_0^x dx\int_{-\sqrt{r^2-x^2}}^{+\sqrt{r^2-x^2}}(r-\sqrt{x^2+y^2})dy,$$

$$=\frac{h}{r}\int_0^x\left(r\sqrt{r^2-x^2}-x^2\log\frac{r+\sqrt{r^2-x^2}}{x}\right)dx,$$

$$=\tfrac{2}{3}hx\sqrt{r^2-x^2}+\tfrac{1}{3}r^2h\sin^{-1}(x/r)-\tfrac{1}{3}\frac{hx^3}{r}\log\frac{r+\sqrt{r^2-x^2}}{x}.$$

Substituting $x=(r/2)\sqrt{2}$, we have for the volume $COFGD$ the expression $\frac{r^2h}{12}[4+\pi-\sqrt{2}.\log(\sqrt{2}+1)]$, and for that of $B-FGD$ $\frac{r^2h}{12}[\pi-4+\sqrt{2}.\log(\sqrt{2}+1)]$

II Solution. Let HK be a circle parallel to AB cutting the hyperbola FDK in the points L and M, and let the diameter HK cut the axis DE at Q. Put $OE=b$, $OF=r$, $CO=h$, $DQ=x$, $LQ=y$. We find from the geometry of the figure $y^2=(2br/x)x+(r^2/h^2)x^2$ as the equation of the hyperbola FDK.

$\therefore$ Area of $FDK=2\int dx\sqrt{\frac{2br}{h}x+\frac{r^2}{h^2}x^2}$ between the limits O,

and $DE=\frac{(r-b)h}{r}$. Integrating we find for this area the expression,

$$\frac{h}{b}\left(r\sqrt{r^2-b^2}-b^2\log\frac{r+\sqrt{r^2-b^2}}{b}\right).$$

$$\therefore\ \text{Volume of } COFGD=\frac{h}{b}\int_0^b\left(r\sqrt{r^2-b^2}=b^2\log\frac{r+\sqrt{r^2-b^2}}{b}\right)db$$

$$=\tfrac{2}{3}hb\sqrt{r^2-b^2}+\tfrac{1}{3}r^2h\sin^{-1}\frac{b}{r}-\tfrac{1}{3}\frac{hb^3}{r}\log\frac{r+\sqrt{r^2-b^2}}{b}\Big),$$

and volume of $BFGD=\frac{1}{3}r^2h\cos^{-1}\dfrac{b}{r}-\frac{2}{3}hb\sqrt{r^2-b^2}+\frac{1}{3}\dfrac{hb^3}{r}\log\dfrac{r+\sqrt{r^2-b^2}}{b}$.

III Solution. Let HK be a circle parallel to AB, and N its centre. Through N draw a diameter parallel to the hyperbolic section FGD. Put $CN=x$, $OE=b$, $BO=r$, $CO=h$, then the area of the circular segment lying between the diameter through N and the parallel chord LM

$$=\frac{r^2x^2}{h^2}\sin^{-1}\frac{bh}{rx}+\frac{br}{h}\sqrt{x^2-\frac{b^2h^2}{r^2}}.$$

$\therefore$ Volume of conical section $COFGD$

$$=\frac{r}{h}\left\{\frac{r}{h}\int x^2\sin^{-1}\frac{bh}{rx}+b\int dx\sqrt{x^2-\frac{b^2h^2}{r^2}}\right\},$$

the integrals to be taken between $h-DE=bh/r$ and h. Thus we find for this volume the expression

$$\tfrac{2}{3}hb\sqrt{r^2-b^2}+\tfrac{1}{3}r^2h\sin^{-1}\frac{b}{r}-\tfrac{1}{3}\frac{hb^3}{r}\log\frac{r+\sqrt{r^2-b^2}}{r};$$

and for the volume of the conical section $DBFG$,

$$\tfrac{1}{3}r^2h\cos^{-1}\frac{b}{r}-\tfrac{2}{3}hb\sqrt{r^2-b^2}+\tfrac{1}{3}\frac{hb^3}{r}\log\frac{r+\sqrt{r^2-b^2}}{r}.$$

Historical Note. The famous astronomer Kepler tried hard to find the volume of such conical sections as the above, but all his efforts proved futile.

Also solved by *GEORGE LILLEY, Ph. D., LL. D.*, and *CHARLES C. CROSS*. Dr. Lilley obtained a numerical result of 6.771 cubic inches, and Professor Cross obtained 2.256979 cubic inches.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

87. Proposed by E. W. MORRELL, A. M., Professor of Mathematics, Montpelier Seminary, Montpelier, Vt.

A and B set out from the same place, and in the same direction. A travels uniformly 18 miles per day, and after 9 days turns and goes back as far as B has traveled during those 9 days; he then turns again, and, pursuing his journey, overtakes B 22½ days after the time they first set out. It is required to find the rate at which B uniformly traveled. [From *Greenleaf's Arithmetic*.]

88. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics, Curry University, Pittsburg, Pa.

Find the principal of a note given March 19, 1891, bearing interest at 6%. Payments: September 1, 1892, \$243.50; January 19, 1893, \$6.90; April 13, 1894, \$19.10; September 19, 1894, \$110.90. Amount due February 22, 1897, \$229.10.

ALGEBRA.

81. Show that $\dfrac{a_1^{\,r}}{(a_1-a_2)(a_1-a_3)(a_1-a_4)\ldots(a_1-a_n)}$

$$+\frac{a_2^{\,r}}{(a_2-a_1)(a_2-a_3)\ldots(a_2-a_n)}+\ldots\frac{a_n^{\,r}}{(a_n-a_1)(a_n-a_2)\ldots(a_n-a_{n-1})}$$

is zero if r is less than $n-1$; to 1 if $r=n-1$, and to $a_1+a_2+{}_3+\ldots a_n$ if $r=n$.

[*C. Smith's Treatise on Algebra.*]

82. $\left.\begin{array}{l} y^2+yz+z^2=a^2 \\ z^2+zx+x^2=b^2 \\ x^2+xy+y^2=c^2 \end{array}\right\}$ find x, y, and z.

[*Ibid.*]

GEOMETRY.

83. **Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.**

θ being variable, find the locus of a point whose coördinates are

$$a\tan(\theta+\alpha),\quad b\tan(\theta+\beta).$$

84. **Proposed by FREDERICK R. HONEY, Ph. B., New Haven, Conn.**

Find the locus of a point which will trisect all arcs having a common chord.

85. **Proposed by S. F. NORRIS, Professor of Astronomy and Mathematics, Baltimore City College, Baltimore, Md.**

Prove by pure geometry. Give direct proof, if possible.

If the bisectors of two angles of a triangle are equal, the triangle is ososceles.

[From *Wentworth's Plane Geometry*, exercise 43, page 72.]

MECHANICS.

61. **Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.**

A body is suspended from a fixed point by an elastic string, which is stretched to double its natural length when the body is in equilibrium. Find how much the body must be depressed, so that when let go, it may just reach the point of suspension.

62. **Proposed by J. SCHEFFER, A. M., Hagerstown, Md.**

A particle of mass m moves in the circumference of an ellipse with constant rate v. It is constrained to move in that circumference by attractive forces in the two foci. To determine the magnitude of these forces.

DIOPHANTINE ANALYSIS.

58. **Proposed by E. S. LOOMIS, Ph. D., Professor of Mathematics in Cleveland West High Sc5ool, Berea, O.**

"The base of a right-angled triangle is 105; find all the perpendiculars and hypotenuses to fit it, such that their values shall be integers."

59. **Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.**

Find the sum of the mth powers of all the numbers less than P and prime to it, and then by substitution find the sume when $m=1, 2, 3, 4, 5$.

NOTES.

THE IRVING HOPKINS FALLACY.

After my having so recently pointed out in THE AMERICAN MATHEMATICAL MONTHLY (Vol. III., pages 122-123) the fallacy of Professor G. C. Edwards of the University of California in his Elements of Geometry in treating parallells, and in *Science* (N. S. Vol. VI. page 491) the gross blunder made by Andrew W. Phillips and Irving Fisher, professors in Yale University, in their Elements of Geometry, could it have been supposed that so respectable a person as Irving Hopkins would deliberately have published the extended fallacy which has just appeared in THE AMERICAN MATHEMATICAL MONTHLY (Vol. IV., pages 251-255) under the ambitious title "Euclidean Geometry without Disputed Axioms"?

It is a simple *petitio principii*. The question is begged in his Proposition IV, which explicitly uses Euclid, III. 31. If any one will turn to III. 31 in any Euclid they will find it proved by Euclid, I. 32. But Euclid I. 32 is the famous angle-sum proposition, which since 1733 has been known to be equivalent to the parallel-postulate, the most disputed of all axioms.

In THE AMERICAN MATHEMATICAL MONTHLY's serial *Non-Euclidean Geometry*, (Vol. I., page 346) is given the Proposition : In any right-angled triangle the two acute angles remaining are taken together equal to one right angle, in the hypothesis of right angle ; greater than one right angle, in the hypothesis of obtuse angle ; but less in the hypothesis of acute angle. In other words, if the angle inscribed in a semicircle is right the geometry is Euclidean ; if obtuse, Riemannian ; if acute, Lobachevskian. GEORGE BRUCE HALSTED.

Austin, Texas.

There are several errors in Mr. Hopkin's paper on "Euclidean Geometry Without Disputed Axioms," but one is enough to which to call attention. In several places he uses Euclid III., 31, which depends upon I., 32, which depends upon I., 29, which depends upon Axiom 12 !

When will we cease trying to accomplish what the masters have found to be impossible ? BENJ. F. YANNEY.

Mt. Union College, Alliance, Ohio.

NOTE ON DR. LILLEY'S ARTICLE IN THE OCTOBER NUMBER.

There is one statement in Professor Lilley's article in the October number about which I wish to say a few words. Concerning the quotient 0. I define division thus : Having given the product of two factors, and one of the factors, to find the other factor.

Thus, the product of two factors$=12$,

One of the factors$=0$,

The other factor$=0$, (Lilley).

Hence $0\times0=12$. Do you believe it ?

Take this illustration—

$$
\begin{aligned}
&12\div3=4\\
&12\div2=6\\
&12\div1=12\\
&12\div.1=120\\
&12\div.01=1200\\
&12\div.001=12000\\
&12\div.0001=120000\\
&\quad\vdots \qquad\quad \vdots\\
&12\div-.0001=-120000\\
&12\div-.001=-12000\\
&12\div-.01=-1200\\
&12\div-.1=-120\\
&12\div-1=-12\\
&12\div-2=-6\\
&12\div-3=-4
\end{aligned}
$$

Here the dividend is constant. The divisor varying continuously, suppose, changes sign in passing through zero (absolute) and at the same time the quotient changes sign in passing through infinity.

The following definition of division may assist in reaching a conclusion : Division is the process of finding how many times a number may be subtracted from another without changing the sign of the remainder.

Apply this definition thus : How many times may zero (absolute) be subtracted from 12 without changing the sign of the remainder.

The answer is, an infinity of infinities, rather than zero.

MILTON L. COMSTOCK.

Knox College, Galesburg, Ill.

Upon some of the points about which I shall disagree with Dr. Lilley he can quote in his favor some of the most brilliant mathematicians that the world has produced. Nevertheless I shall endeavor to show that they and he have failed to take a common sense view of the subject. Upon one point I think I am safe in saying that the Doctor's position is unique. The source of his errors lies, in my opinion, in his conception of infinity and zero.

Concerning the former he says : "If 12/⓪=infinity or *the largest possible number*," etc.

From this I can not but infer that he thinks infinity is a constant and that that constant is the largest possible number. He says "12/⓪=⊖, where ⊖ represents no number of times." Again we infer that he believes that while ⊖ represents *no number of times*, ∞ must represent *some number of times*.

He uses too many zeros. He has ⓪=absolute zero, ⓪=*no number of times*, and ◎=an infinitesimal. He refers to the latter zero as follows : It is a consequence of confounding the 0 arising from dividing a by infinity, with the absolute zero, that so much confusion has arisen."

He has the authority of Davies and Peck's Mathematical Dictionary for this statement, but this does not make it true. Nothing could be more confusing to the average man of common sense than the Doctor's three zeros.

I have no use for more than one. My mind is perfectly clear as to what that is but it is not so clear as to what ∞ is. It is much easier to tell what ∞ is not than what it is.

If we suppose $a/h=N$, where h is a very small positive quantity, then N is a very large one. As h grows smaller and smaller, N grows larger and larger, but N will *not* become infinite *so long as h has the smallest shadow of value.* So long as h has the slightest value we can form some conception of the value of N. It is only when h becomes equal to 0 that N suddenly swings clear out of our powers of conception. It is then, and then only, that it becomes infinite.

I must dissent from even so great a mathematician as Professor De Morgan when he said that he dated his first clear conception of mathematical infinity from the time when he rejected the relation $a/0=\infty$.

The very fact that he had a clear conception of what he called infinity proved that it was not the real infinity.

I have no criticism to make on Dr. Lilley's disposition of 0/0. I would have liked it better if he had added Art. 175 of his Higher Algebra, which reads: "The symbol 0/0 does not always mean *indetermination.* It is often the result of a particular condition which makes a factor, common to both terms of a fraction become zero. Thus," etc. Here follows the well known illustration by using $\dfrac{a^2-x^2}{a-x}=a+x$.

He does not find it necessary to introduce the infinitesimal to prove that the expression equals $2a$ when $x=a$, as do many writers on the differential calculus when discussing the expression $\dfrac{y_1-y}{x_1-x}$. In this he is right, for if $a-x$ were an infinitesimal the value of the fraction would differ from $2a$ by an infinitesimal, and *an equation that differs from the truth by an infinitesimal is not true at all.*

HENRY HEATON.

Atlantic, Iowa.

We see no place for confusion in the use of the symbols 0 and ∞, and, therefore, of course, no necessity of introducing new symbols to avoid confusion. If 0 is a symbol used to denote the absence of quantity, and ∞ to denote a quantity larger than any assignable quantity however large, then all operations with these symbols are meaningless. For example, $5\div\infty$, $0\div5$, $5\div0$, $0\div0$, $0\times\infty$, etc., are impossible operations. Standing apart from conditions imposed upon quantities from which these symbols arise by certain limitations, they have no meaning whatever. Hence, when these symbols do arise in mathematical investigations, they must be interpreted in conformity to fundamental principles and conceptions. When we say that $5\div\infty=0$, we mean that the limit of $5\div$ a quantity which increases indefinitely $=0$, concisely expressed thus $\lim_{h\doteq\infty}\left[\dfrac{5}{h}\right]=0$.

This is an absolutely accurate statement. 0 is the absolute zero and not an *infinitesimal.* In like manner $5 \div 0 = \infty$ is an abbreviated and inaccurate expression for the following: 5 divided by a quantity which decreases indefinitely gives a quotient larger than any quantity however large, or briefly and *accurately* thus $\lim_{e \doteq 0} \left[\frac{5}{e} \right] = \infty$.

Discussion on a subject of this sort is trivial, but if it results in giving clearer notions of the use of 0 and ∞, a good work will have been done.

B. F. F.

BOOKS AND PERIODICALS.

Plane and Solid Analytical Geometry. By Frederick H. Bailey, A. M. (Harvard), and Frederick S. Woods, Ph. D. (Göttengen), Assistant Professors of Mathematics in the Massachusetts Institute of Technology. 8vo. Cloth, 371 pages. Boston and Chicago: Ginn & Co.

Besides the usual subjects treated in the ordinary text-books of Analytical Geometry, the following additional ones are treated with sufficient fullness to give a student a fair knowledge of them, viz: Radical Axis, and Properties of Pole and Polars. More attention should be given to these subjects in the future by the ordinary student. Besides deriving the equations of the conics in the usual way, the authors have also derived the equations by passing a plane through a right circular cone, thus emphasizing the relation of the geometrical to the analytical method of treatment. About seventy pages are given to the treatment of Solid Analytical Geometry. The treatment here is clear and concise, affording the student an excellent introduction to this important subject. B. F. F.

Famous Problems of Elementary Geometry.—The Duplication of the Cube; The Trisection of an Angle; and The Quadrature of the Circle. Authorized translation Vorträge Ueber Ausgewälte Fragen der Elementargeometrie Ausgearbeitet von F. Tägert. By Wooster Woodruff Beman, Professor of Mathematics in the University of Michigan, and David Eugene Smith, Professor of Mathematics in the Michigan State Normal College. 8vo. Cloth, 80 pages. Price, 55 cents. Boston and Chicago: Ginn & Co.

This book deals with the possibility of elementary geometric constructions in general, the nature of transcendental numbers, and with the transcendence of e and π. While no knowledge of the calculus is needed to read this book, the calculus not being employed in any of the discussions, yet a fair knowledge of the theory of equations and series is absolutely necessary to make it easy reading. The translators deserve the thanks of students and teachers of mathematics, and for putting out books of such scientific value at a very reasonable price, the publishers should receive encouragement by a large sale of this book.

B. F. F.

Popular Scientific Lectures. By Ernst Mach, formerly Professor of Physics in the University of Prague, now Professor of the History and Theory of Inductive Science in the University of Vienna. Translated by Thomas J. McCormack. Second Edition, Revised and Enlarged. 8vo. Cloth, 382 pages. Price, $1.00. Chicago: The Open Court Publishing Co.

These sixteen lectures on various scientific subjects are full of interest to all classes of readers. The lecture "On the Relative Educational Value of the Classics and the Mathematico-Physical Sciences" is especially interesting, and is a fair exposition of the argument *pro* and *con*.

In acquiring an education two things are requisite: first, the development of thought, and second, the power to express thought in a clear and forcible manner. The first is gained by the study of mathematics and the natural sciences, the second, by the classics. Hence, in securing the most symmetrical and stable development of the mind, it is essential that the student pursue his study in the classics, especially Latin, as well as mathematics and the natural sciences. Dr. Mach makes this very pertinent statement: "Here I may count upon assent when I say that mathematics and the natural sciences pursued alone as means of instruction yield a richer education, an education in matter and form, a more general education, an education better adapted to the needs and spirit of the times, than the philological branches pursued alone would yield." In bringing out the translation of these valuable lectures, the translator has the thanks of English readers.

B. F. F.

Field-Manual for Railroad Engineers. By J. C. Nagle, M. A., M. C. E., Professor of Civil Engineering in the Agricultural and Mechanical College of Texas. $4\frac{1}{2}$x$6\frac{3}{4}$ inches, Flexible Morocco, xv+394 pages. Price, $2.50. New York: John Wiley & Sons.

This book is in every way a model field-manual. It contains six chapters. Chapter I.—Reconnoisance: Chapter II.—Preliminary Surveys: Chapter III.—Location, Art. 7, Projecting Location; Art. 8, Simple Curves; Art. 9, Compound Curves; Art. 10, Track Problems: Chapter IV.—Transition Curves; Art. 11, Theory of the Transition Curve; Art. 12, Field Work; Art. 13, Transition Curve Problems: Chapter V.—Frogs and Switches; Art. 14, Turnouts; Art. 15, Crossovers; Art. 16, Crossing-Frogs and Crossing-Slips: Chapter VI.—Construction; Art. 17, Definitions, General Consideration, Vertical Curves, Elevation of Outer Rail; Art. 18, Earthworks; Art. 19, Grade and Ballast Stakes, Culverts, Bridges, and Tunnels; Art. 20, Monthly and Final Estimates.

The above abridged outline of the table of contents indicates very imperfectly the scope and character of this work. In it may be found the most essential things to be known in civil engineering discussed in a way not only that may be understood, but that can be easily understood, by any one familiar with algebra, geometry, and trigonometry.

B. F. F.

A Chapter in the History of Mathematics. An Address by Vice President W. W. Beman, Chairman of Section A, before the Section of Mathematics and Astronomy, American Association for the Advancement of Science, Detroit Meeting, August, 1897. Pamphlet, 20 pages.

In this very able address by Professor Beman is gathered together some valuable history concerning the introduction in mathematics of the square root of negative numbers. The address bears evidence of careful research, and is of great interest to all who are concerned about the progress and development of that great body of doctrine known as mathematics.

B. F. F.

Darwin and After Darwin: Part II. Post-Darwinian Questions. Heredity and Utility 8vo. Cloth, xii and 344 pages. Price, $1.50. With portrait of Romanes. Chicago: The Open Court Publishing Co.

This, as all of Dr. Romanes' works, bears the evident marks of a profound thinker and scholar. The volume before us is chiefly devoted to a consideration of those Post-Darwinian theories which involve fundamental questions of Heredity and Utility, and contains the most valuable results of a deep study of the evolutionary problem. B. F. F.

The Probability of Hit when the Probable Error in Aim is Known with a Comparison of the Probabilities of Hit by the Method of Independent and Parallel Fires from Mortar Batteries. By Mansfield Merriman, Professor of Civil Engineering in Lehigh University. Pamphlet, 12 pages. Reprinted from the Journal of the U. S. Artillery, Vol. VIII, No. 2.

The problem considered in this paper is, To find the probability of hit on the target or deck of a ship whose area is $4aA$, where $2a$ is the width of the target in azimuth and $2A$ its length in range, a shot being fired with the intention of hitting the center. B. F. F.

Contributions to the Geometry of the Triangle. By Robert J. Aley, A. M , Professor of Mathematics in the University of Indiana. Pamphlet, 32 pages.

This thesis was accepted by the Department of Mathematics of the University of Pennsylvania in partial fulfillment of the requirements for the degree of Doctor of Philosophy, which is a sufficient testimonial of its importance and value. B. F. F.

Periodico di Mathematica Per L'Insegnamento Secondario. Dott. G. Lazzeri. November-December number.

The Mathematical Gazette. Edited by F. S. Macauley, M. A., D. Sc. October number.

Bollettino della Associazione "Mathesis" Fra Gl'Insignanti di Mathematica delle Scuole Medie.

Revue Semestrielle des Publications Mathomatiques Rédigée sous les auspices de la Sociéte Mathematique d'Amsterdam. Par P. H. Schoute, D. J. Korteweg, J. C. Kluyver, W. Kapteyn, P. Zeeman.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.00 per year in advance. Single numbers, 25 cents. The American Monthly Review of Reviews Co., 13 Astor Place, New York.

The December number of the *American Monthly Review of Reviews* has several interesting features. Mr. Ernest Knaufft, editor of the *Art Student*, contributes an elaborate study of "John Gilbert and Illustration in the Victorian Era"; Dr. Clifton H. Levy tells "How the Bible Came Down to Us," with a number of reproductions from ancient Biblical manuscripts and printed texts; Lady Henry Somerset pays a tribute to the late Duchess of Teck; an English officer in the Indian service writes about the Ameer of Afghanistan; Mr. E. V. Smalley discusses Canadian reciprocity, and Mr. Alex. D. Anderson summarizes the progress of the American Republics. There is also a 23-page illustrated department devoted to the season's new books, with an introductory chapter, by Albert Shaw, on "Some American Novels and Novelists." Altogether, the *Review* is not lacking in novelty or variety. B. F. F.

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THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. IV. DECEMBER, 1897. No. 12.

BIOGRAPHY.

LEONHARD EULER.

BY B. F. FINKEL.

LEONHARD EULER (oi'ler), one of the greatest and most prolific mathematicians that the world has produced, was born at Basel, Switzerland, on the 15th day of April, 1707, and died at St. Petersburg, Russia, November the 18th (N. S.), 1783. Euler received his preliminary instruction in mathematics from his father who had considerable attainments as a mathematician, and who was a Calvinistic* pastor of the village of Riechen, which is not far from Basel. He was then sent to the University of Basel where he studied mathematics under the direction of John Bernoulli, with whose two sons, Daniel and Nicholas, he formed a life-long friendship. Geometry soon became his favorite study. His genius for analytical science soon gained for him a high place in the esteem of his instructor, John Bernoulli, who was at the time one of the first mathematicians of Europe. Having taken his degree as Master of Arts

The Encyclopedia Brittanica says Euler's father was a Calvinistic minister, while W. W. R. Ball, in his *History of Mathematics*, says he was a Lutheran minister. Euler himself was a Calvinist in doctrine, as the following, which is his apology for prayer, indicates: "I remark, first, that when God established the course of the universe, and arranged all the events which must come to pass in it, he paid attention to all the circumstances which should accompany each event; and particularly to the dispositions, to the desires, and prayers of every intelligent being; and that the arrangement of all events was disposed in perfect harmony with all these circumstances. When, therefore, a man addresses God a prayer worthy of being heard it must not be imagined that such a prayer came not to the knowledge of God till the moment it was formed. That prayer was already heard from all all eternity; and if the Father of Mercies deemed it worthy of being answered, he arranged the world expressly in favor of that prayer, so that the accomplishment should be a consequence of the natural course of events. It is thus that God answers the prayers of men without working a miracle."

in 1723, Euler afterwards applied himself, at his father's desire, to the study of theology and the Oriental languages, with the view of entering the ministry, but, with his father's consent, he returned to his favorite pursuit, the study of mathematics. At the same time, by the advice of the younger Bernouillis, who had removed to St. Petersburg in 1725, he applied himself to the study of physiology, to which he made useful applications of his mathematical knowledge ; he also attended the lectures of the most eminent professors of Basel. While he was eagerly engaged in physiological researches, he composed a dissertation on the nature and propagation of sound. In his nineteenth year he also composed a dissertation in answer to a prize-question concerning the masting of ships, for which he received the second prize from the French Academy of Sciences.

When his two close friends, Daniel and Nicholas Bernoulli, went to Russia, they induced Catherine I, in 1727, to invite Euler to St. Petersburg, where Daniel, in 1733, was assigned to the chair of mathematics. Euler took up his residence in St. Petersburg, and was made an associate of the Academy of Sciences. In 1730 he became professor of physics, and in 1733 he succeeded his friend Daniel Bernoulli, who resigned on a plea of ill health.

At the commencement of his astonishing career, he enriched the Academical collection with many memoirs, which excited a noble emulation between him and the Bernouillis, though this did not in any way affect their friendship. It was at this time that he carried the integral calculus to a higher degree of perfection, invented the calculation of sines, reduced analytical operations to greater simplicity, and threw new light on nearly all parts of pure or abstract mathematics. In 1735, an astronomical problem proposed by the Academy, for the solution of which several eminent mathematicians had demanded several months' time, was solved by Euler in three days with the aid of improved methods of his own, but the effort threw him into a fever which endangered his life and deprived him of his right eye, his eyesight having been impaired by the severity of the climate. With still superior methods, this same problem was solved later by the illustrious German mathematician, Gauss.

In 1741, at the request, or rather command, of Frederick the Great, he moved to Berlin, where he was made a member of the Academy of Sciences, and Professor of Mathematics. He enriched the last volume of the *Mélanges* or Miscellanies of Berlin, with five memoirs, and these were followed, with astonishing rapidity, by a great number of important researches, which were scattered throughout the annual memoirs of the Prussian Academy. At the same time, he continued his philosophical contributions to the Academy of St. Petersburg, which granted him a pension in 1742.

The respect in which he was held by the Russians was strikingly shown in 1760, when a farm he occupied near Charlottenburg happened to be pillaged by the invading Russian army. On its being ascertained that the farm belonged to Euler, the general immediately ordered compensation to be paid, and the Empress Elizabeth sent an additional sum of four thousand crowns. The despotism of Anne I. caused Euler, who was a very timid man, to shrink from public

affairs, and to devote all his time to science. After his call to Berlin, the Queen of Prussia who received him kindly, wondered how so distinguished a scholar should be so timid and reticent. Euler replied, "Madam, it is because I come from a country where, when one speaks, one is hanged."

In 1766, Euler, with difficulty, obtained permission from the King of Prussia to return to St. Petersburg, to which he had been originally called by Catherine II. Soon after returning to St. Petersburg a cataract formed in his left eye, which ultimately deprived him of sight, but this did not stop his wonderful literary productiveness, which continued for seventeen years—until the day of his death. It was under these circumstances that he dictated to his amanuensis, a tailor's apprentice who was absolutely devoid of mathematical knowledge, his *Anleitung zur Algebra*, or *Elements of Algebra*, 1770, a work which, though purely elementary, displays the mathematical genius of its author, and is still considered one of the best works of its class. Euler was one of the very few great mathematicians who did not deem it beneath the dignity of genius to give some attention to the recasting of elementary processes and the perfecting of elementary text-books, and it is not improbable that modern mathematics is as greatly indebted to him for his work along this line as for his original creative work.

Another task to which he set himself soon after returning to St. Petersburg was the preparation of his *Lettres à une Princesse d'Allemagne sur quelques sujects de Physique*, (3 vols. 1768-72). These letters were written at the request of the princess of Anhalt-Dessau, and contain an admirably clear exposition of the principal facts of mechanics, optics, acoustics, and physical astronomy. Theory, however, is frequently unsoundly applied in it, and it is to be observed generally that Euler's strength lay rather in pure than in applied mathematics. In 1755, Euler had been elected a foreign member of the Academy of Sciences at Paris, and sometime afterwards the academical prize was adjudged to three of his memoirs *Concerning the Inequalities in the Motions of the Planets*. The two prize-problems proposed by the same Academy in 1770 and 1772 were designed to obtain a more perfect theory of the moon's motion. Euler, assisted by his eldest son, Johann Albert, was a competitor for these prizes and obtained both. In his second memoir, he reserved for further consideration the several inequalities of the moon's motion, which he could not determine in his first theory on account of the complicated calculations in which the method he then employed had engaged him. He afterward reviewed his whole theory with the assistance of his son and Krafft and Lexell, and pursued his researches until he had constructed the new tables, which appeared with the great work in 1772. Instead of confining himself, as before, to the fruitless integration of three differential equations of the second degree, which are furnished by mathematical principles, he reduced them to three ordinates which determine the place of the moon ; and he divides into classes all the inequalities of that planet, as far as they depend either on the elongation of the sun and moon, or upon the eccentricity, or the parallax, or the inclination of the lunar orbit. The inherent difficulties of this

task were immensely enhanced by the fact that Euler was virtually blind, and had to carry all the elaborate computations involved in his memory. A further difficulty arose from the burning of his house and the destruction of a greater part of his property in 1771. His manuscripts were fortunately preserved. His own life only was saved by the courage of a native of Basel, Peter Grimmon, who carried him out of the burning house.

Some time after this, the celebrated Wenzell, by couching the cataract, restored his sight ; but a too harsh use of the recovered faculty, together with some carelessness on the part the surgeons, brought about a relapse. With the assistance of his sons, and of Krafft and Lexell, however, he continued his labors, neither the loss of his sight nor the infirmities of an advanced age being sufficient to check his activity. Having engaged to furnish the Academy of St. Petersburg with as many memoirs as would be sufficient to complete its acts for twenty years after his death, he in seven years transmitted to the Academy above seventy memoirs, and left above two hundred more, which were revised and completed by another hand.

Euler's knowledge was more general than might have been expected in one who had pursued with such unremitting ardor, mathematics and astronomy, as his favorite studies. He had made considerable progress in medicine, botany, and chemistry, and he was an excellent classical scholar and extensively read in general literature. He could repeat the *Ænied* of Virgil from the beginning to the end without hesitation, and indicate the first and last line of every page of the edition which he used. But such lines from Virgil as, "The anchor drops, the rushing keel is staid," always suggested to him a problem and he could not help enquiring what would be the ship's motion in such a case.

Euler's constitution was uncommonly vigorous and his general health was always good. He was enabled to continue his labors to the very close of his life so that it was said of him, that he ceased to calculate and to breath at nearly the same moment. His last subject of investigation was the motions of balloons, and the last subject on which he conversed was the newly discovered planet Herschel.

On the 18th of September, 1783, while he was amusing himself at tea with one of his grandchildren, he was struck with apoplexy, which terminated the illustrious career of this wonderful genius, at the age of seventy-six. His works, if printed in their completeness, would occupy from 60 to 80 quarto volumes. However, no complete edition of Euler's writings has been published, though the work has been begun twice.

He was simple, upright, affectionate, and had a strong religious faith. His single and unselfish devotion to the truth and his joy at the discoveries of science whether made by himself or others, were striking attributes of his character. He was twice married, his second wife being a half-sister of his first, and he had a numerous family, several of whom attained to distinction. His *éloge* was written for the French Academy by Condorcet, and an account of his life, with a list of his works, was written by Von Fuss, the secretary of the Imperial Academy of St. Petersburg.

As has been said, Euler wrote an immense number of works, chief of which are the following: *Introductio in Analysin infinitorum*, 1748, which was intended to serve as an introduction to pure analytical mathematics. This work produced a revolution in analytical mathematics, as the subject of which it treated had hitherto never been presented in so general and systematic a manner. The first part of the Analysis Infinitorum contains the bulk of the matter which is to be found in modern text-books on algebra, theory of equations, and trigonometry. In the algebra, he paid particular attention to the expansion of various functions in series, and to the summation of given series; and pointed out explicitly that an infinite series can not be safely employed in mathematical investigations unless it is convergent. In trigonometry, he introduced (simultaneously with Thomas Simpson in England) the now current abbreviations for trigonometric functions, and simplified formulæ by the simple expedient of designating the angles of a triangle by A, B, C, and the opposite sides by a, b, c. He also showed that the trigonometrical and exponential functions are connected by the the relation $\cos\theta + i\sin\theta = e^{i\theta}$. Here too we meet the symbol e used to denote the base of the Naperian logarithms, namely the incommensurable number 2.7182818 . . . and the symbol π used to denote the incommensurable number 3.14159265 . . . The use of a single symbol to denote the number 2.7182818 . . . seemes to be due to Cotes, who denoted it by M. Newton was probably the first to employ the literal exponential notation, and Euler using the form a^z, had taken a as the base of any system of logarithms. It is probable that the choice of e for a particular base was determined by its being the vowel consecutive to a, or, still more probable because e is the initial of the word *exponent*.

The use of a single symbol to denote 3.14159265 . . . appears to have been introduced by John Bournilli, who represented it by c. Euler in 1734 denoted it by p, and in a letter of 1736 in which he enunciated the theorem that the sum of the square of the reciprocals of the natural numbers is $\frac{1}{6}\pi^2$, he uses the letter c. Chr. Goldbach in 1742 used π, and after the publication of Euler's Analysis, the sympol π was generally employed, the choice of π being determined by the initial of the word, *περιφέρεια*—*periphereia*.

The second part of the *Analysis Infinitorum* is on analytical geometry. Euler begins this part by dividing curves into algebraic and transcendental, and establishes a number of propositions which are true for all algebraic curves. He then applied these to the general equation of the second degree in two dimensions, showed that it represents the various conic sections, and deduces most of their properties from the general equation. He also considered the classification of cubic, quartic, and other algebraic curves. He next discussed the question as to what surfaces are represented by the general equation of the second degree in three dimensions, and how they may be discriminated one from the other. Some of these surfaces had not been previously investigated. In this work he also laid down the rules for the transformation of coördinates in space. Here also we find the first attempt to bring the curvature of surfaces within the domain of mathematics, and the first complete discussion of tortuous curves.

In 1755 appeared *Institutiones Calculi Differentialis*, to which the *Analysis Infinitorum* was intended as an introduction. This is the first text-book on the differential calculus which has any claim to be regarded as complete, and it may be said that most modern treatises on the subject are based upon it.

At the same time, the exposition of the principles of the subject is often prolix and obscure, and sometimes not quite accurate.

This series of works was completed by the publication in three volumes in 1768 to 1770 of the *Institutiones Calculi Integralis*, in which the results of several of Euler's earlier memoirs on the same subjects and on differential equations are included. In this treatise as in the one on the differential calculus was summed up all that was at that time known on the subject. The beta and gamma functions were invented by Euler, and are discussed here, but only as methods of reduction and integration. His treatment of elliptic integrals is superficial. The classic problems on isoperimetrical curves, the brachistochrone in a resisting medium, and theory of geodesics had engaged Euler's attention at an early date, and the solving of which led him to the calculus of variations. The general idea of this was laid down in his *Curvarum Maximi Minimive Proprietate Gaudentium Inventio Nova ac Facilis*, published in 1744, but the complete development of the new calculus was first effected by Lagrange in 1759 The method used by Lagrange is described in Euler's integral calculus, and is the same as that given in most modern text-books on the subject.

In 1770, Euler published the *Anleitung zur Algebra* in two volumes. The first volume treats of determinate algebra. This work includes the proof of the binomial theorem for any index, which is still known by Euler's name. The proof, which is not accurate according to the modern views of infinite series, depends upon the principle of the permanence of equivalent forms, and may be seen in C. Smith's *Treatise on Algebra*, pages 336-7. Euler's proof with important additions due to Cauchy, may be seen in G. Chrystal's Algebra, Part II.

It is a fact worthy of note that Euler made no attempt to investigate the convergency of the series, though he clearly recognized the necessity of considering the convergency of infinite series. While Euler recognized the convergency of series, his conclusions in reference to infinite series are not always sound. In his time no clear notion as to what constitutes a convergent series existed, and the rigid treatment to which infinite series are now subjected was undreamed of. Euler concluded that the sum of the oscillating series $1-1+1-1+1-1+\ldots=\frac{1}{2}$, for the reason, that by stopping with an even number of terms the sum is 0, and by stopping with an odd number of terms the sum is 1. Hence, the sum of the series is $\frac{1}{2}(0+1)=\frac{1}{2}$. Guido Grandi went so far as to conclude that $\frac{1}{2}=0+0+0+0\ldots$ The paper in which Euler cautions against divergent series contains the proof that $\ldots\frac{1}{n^2}+\frac{1}{n}+1+n+n^2+n^3\ldots=0$. His proof is as follows, $n+n^2+n^3+\ldots=\frac{n}{1-n}$, $1+\frac{1}{n}+\frac{1}{n^2}+\ldots=\frac{n}{n-1}$. $\frac{n}{n-1}+\frac{n}{1-n}=0$. Euler had no hesitation in writing $1-3+5-7+9-\ldots=0$, and he confidently believed that $\sin\phi-2\sin 2\phi+3\sin 3\phi-\ldots=0$.

A remarkable development, due to Euler, is what he named the hypergeometrical series, the summation of which he observed to be dependent upon the integration of linear differential equations of the second order, but it remained for Gauss to point out that for special values of the letters, this series represented nearly all the functions then known. By giving the factors 641×6700417 of the number $2^{2^n}+1=4294967297$ when $n=5$, he pointed out the fact that this expression did not always represent primes, as was supposed by Fermat.

The sources from which this biography has been obtained are Cajori's and Ball's *History of Mathematics*, and the *Encyclopedia Britannica*.

MOMENTS OF INERTIA.

By G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

It is the purpose of this paper to put on record formulæ for the Moments of Inertia of the plane areas, $\left(\frac{x}{a}\right)^{\frac{2}{2m+1}}+\left(\frac{y}{b}\right)^{\frac{2}{2n+1}}=1$, and the solid bounded by the surface, $\left(\frac{x}{a}\right)^{\frac{2}{2m+1}}+\left(\frac{y}{b}\right)^{\frac{2}{2n+1}}+\left(\frac{z}{c}\right)^{\frac{2}{2p+1}}=1$.

Let μ be the mass of a unit, (a) area, (b) volume.

(a) Areas, when n and m are positive integers.

For the x-axis,

$$I=4\mu\int\int y^2\,dx\,dy=\frac{4ab^3\mu}{\frac{4}{(2m+1)(2n+1)}}\cdot\frac{\Gamma(m+\frac{1}{2})\Gamma(3n+\frac{3}{2})}{\Gamma(m+3n+3)}$$

$$=\frac{\mu ab^3(2m+1)(2n+1)(6n+1)\Gamma(m+\frac{1}{2})\Gamma(3n+\frac{1}{2})}{2(m+3n+2)(m+3n)(m+3n+1)\Gamma(m+3n)}\ldots\ldots\ldots\ldots(1)$$

$$=\frac{1.3.5\ldots\ldots(2m+1)\times1.3.5\ldots\ldots(6n+1)}{2.4.6\ldots\ldots2(m+3n+2)}\cdot2\pi\mu ab^3(2n+1)\ldots\ldots(2).$$

For the y-axis,

$$I_1=4\mu\int\int x^2\,dx\,dy=\frac{\mu a^3b(2m+1)(2n+1)(6m+1)\Gamma(3m+\frac{1}{2})\Gamma(n+\frac{1}{2})}{2(3m+n+2)(3m+n+1)(3m+n)\Gamma(3m+n)}\ldots\ldots\ldots(3)$$

$$=\frac{1.3.5\ldots\ldots(6m+1)\times1.3.5\ldots\ldots(2n+1)}{2.4.6\ldots\ldots2(3m+n+2)}\cdot2\pi\mu a^3b(2m+1)\ldots\ldots(4).$$

For an axis through its center perpendicular to its plane,

$$I_2=I+I_1\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

The product of inertia of a quadrant about its axes is,

$$p=\mu\int\int xy\,dx\,dy=\frac{\mu a^2b^2}{\frac{4}{(2m+1)(2n+1)}}\cdot\frac{\Gamma(2m+2)\Gamma(2n+1)}{\Gamma(2m+2n+3)}$$

$$=\frac{\mu a^2b^2mn(2m+1)(2n+1)\Gamma(2m)\Gamma(2n)}{4(m+n+1)(m+n)(2m+2n+1)\Gamma(2m+2n)}\ldots\ldots\ldots\ldots(6)$$

$$=\frac{1.2.3.4\ldots\ldots(2m+1)\times 1.2.3.4\ldots\ldots(2n+1)}{1.2.3.4\ldots\ldots(2m+2n+2)}\cdot\frac{\mu a^2b^2}{4}\ldots\ldots\ldots(7).$$

Let $m=n=0$. Then for the ellipse, $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$.

$$I=\tfrac{1}{4}\pi\mu ab^3,\quad I_1=\tfrac{1}{4}\pi\mu a^3b,\quad I_2=\tfrac{1}{4}\pi\mu ab(a^2+b^2),\quad p=\tfrac{1}{8}\mu a^2b^2.$$

Let $m=n=1$. Then for the hypocycloid, $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$.

$$I=\tfrac{21}{512}\pi\mu ab^3,\quad I_1=\tfrac{21}{512}\pi\mu a^3b,\quad I_2=\tfrac{21}{512}\pi\mu ab(a^2+b^2),\quad p=\tfrac{1}{80}\mu a^2b^2.$$

(b) Solids, when m, n, and p are positive integers.
With regard to the plane (yz),

$$\mathbf{I}=8\mu\int\int\int x^2\,dx\,dy\,dz$$

$$=\frac{8\mu a^3bc}{\frac{8}{(2m+1)(2n+1)(2p+1)}}\cdot\frac{\Gamma(3m+\frac{3}{2})\Gamma(n+\frac{1}{2})\Gamma(p+\frac{1}{2})}{\Gamma(3m+n+p+\frac{5}{2})}$$

$$=\frac{4\mu a^3bc(2m+1)(2n+1)(2p+1)(6m+1)\Gamma(3m+\frac{1}{2})\Gamma(n+\frac{1}{2})\Gamma(p+\frac{1}{2})}{(6m+2n+2p+5)(6m+2n+2p+3)(6m+2n+2p+1)\Gamma(3m+n+p+\frac{1}{2})}\ldots\ldots(8)$$

$$=\frac{1.3.5\ldots\ldots(6m+1)\times 1.3.5\ldots\ldots(2n+1)\times 1.3.5\ldots\ldots(2p+1)}{1.3.5\ldots\ldots(6m+2n+2p+5)}$$

$$\times 4\mu\pi a^3bc(2m+1)\ldots\ldots(9).$$

With regard to the plane (xz),

$$\mathbf{I}_1=8\mu\int\int\int y^2\,dx\,dy\,dz$$

$$=\frac{4\mu ab^3c(2m+1)(2n+1)(2p+1)(6n+1)\Gamma(m+\frac{1}{2})\Gamma(3n+\frac{1}{2})\Gamma(p+\frac{1}{2})}{(2m+6n+2p+5)(2m+6n+2p+3)(2m+6n+2p+1)\Gamma(m+3n+p+\frac{1}{2})}\ldots(10)$$

$$=\frac{1.3.5\ldots\ldots(2m+1)\times 1.3.5\ldots\ldots(6n+1)\times 1.3.5\ldots\ldots(2p+1)}{1.3.5\ldots\ldots(2m+6n+2p+5)}$$

$$\times 4\mu\pi ab^3c(2n+1)\ldots\ldots(11).$$

With regard to the plane (xy),

$$\mathbf{I}_2=8\mu\int\int\int z^2\,dx\,dy\,dz$$

$$=\frac{4\mu abc^3(2m+1)(2n+1)(2p+1)(6p+1)\Gamma(m+\frac{1}{2})\Gamma(n+\frac{1}{2})\Gamma(3p+\frac{1}{2})}{(2m+2n+6p+5)(2m+2n+6p+3)(2m+2n+6p+1)\Gamma(m+n+3p+\frac{1}{2})}\ldots(12)$$

$$=\frac{1.3.5\ldots\ldots(2m+1)\times 1.3.5\ldots\ldots(2n+1)\times 1.3.5\ldots\ldots(6p+1)}{1.3.5\ldots\ldots(2m+2n+6p+5)}$$

$$\times 4\mu\pi abc^3(2p+1)\ldots\ldots(13).$$

$\mathbf{I}_3=\mathbf{I}+\mathbf{I}_1$, for z-axis, $\mathbf{I}_4=\mathbf{I}+\mathbf{I}_2$, for y-axis,

$\mathbf{I}_5=\mathbf{I}_1+\mathbf{I}_2$, for x-axis, $\mathbf{I}_6=\mathbf{I}+\mathbf{I}_1+\mathbf{I}_2$, for center.

Product of inertia of an octant of the solid with regard to the (y, z) axes.

$$P=\mu\int\int\int yz\,dx\,dy\,dz=\frac{\frac{\mu ab^2c^2}{8}}{(2m+1)(2n+1)(2p+1)}\cdot\frac{\Gamma(m+\frac{1}{2})\Gamma(2n+1)\Gamma(2p+1)}{\Gamma(m+2n+2p+\frac{5}{2})}$$

$$=\frac{4\mu ab^2c^2np(2m+1)(2n+1)(2p+1)\Gamma(m+\frac{1}{2})\Gamma(2n)\Gamma(2p)}{(2m+4n+4p+5)(2m+4n+4p+3)(2m+4n+4p+1)\Gamma(m+2n+2p+\frac{1}{2})}\ldots(14)$$

$$=\frac{1.2.3\ldots\ldots(2n+1)\times 1.2.3\ldots\ldots(2p+1)\times\frac{1}{2}.\frac{3}{2}.\frac{5}{2}\ldots\ldots\left(\frac{2m+1}{2}\right)}{\frac{1}{2}.\frac{3}{2}.\frac{5}{2}.\frac{7}{2}\ldots\ldots\left(\frac{2m+4n+4p+5}{2}\right)}\cdot\frac{\mu ab^2c^2}{4}$$

$$=\frac{1.2.3.4\ldots\ldots(2n+1)\times 1.2.3.4\ldots\ldots(2p+1)}{(2m+3)(2m+5)\ldots\ldots(2m+4n+4p+5)}\cdot\mu ab^2c^2.2^{2(n+p)}\ldots(15).$$

With regard to the axes (x, z),

$$P_1=\mu\int\int\int xz\,dx\,dy\,dz$$

$$=\frac{4\mu a^2bc^2mp(2m+1)(2n+1)(2p+1)\Gamma(2m)\Gamma(n+\frac{1}{2})\Gamma(2p)}{(4m+2n+4p+5)(4m+2n+4p+3)(4m+2n+4p+1)\Gamma(2m+n+2p+\frac{1}{2})}\ldots(16)$$

$$=\frac{2^{2(m+p)}1.2.3.4\ldots\ldots(2m+1)\times 1.2.3.4\ldots\ldots(2p+1)}{(2n+3)(2n+5)\ldots\ldots(4m+2n+4p+5)}\mu a^2bc^2\ldots(17).$$

With regard to the axes (x, y),

$$P_2=\mu\int\int\int xy\,dx\,dy\,dz$$

$$=\frac{4\mu a^2b^2cmn(2m+1)(2n+1)(2p+1)\Gamma(2m)\Gamma(2n)\Gamma(p+\frac{1}{2})}{(4m+4n+2p+5)(4m+4n+2p+3)(4m+4n+2p+1)\Gamma(2m+2n+p+\frac{1}{2})}\ldots(18)$$

$$=\frac{2^{2(m+n)}1.2.3.4\ldots\ldots(2m+1)\times 1.2.3.4\ldots\ldots(2n+1)}{(2p+3)(2p+5)\ldots\ldots(4m+4n+2p+5)}\cdot\mu a^2b^2c\ldots(19).$$

Let $m=n=p=0$. Then for $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$,

$\mathbf{I}=\frac{4}{15}\mu\pi a^3bc$, $\mathbf{I}_1=\frac{4}{15}\mu\pi ab^3c$, $\mathbf{I}_2=\frac{4}{15}\mu\pi abc^3$,

$\mathbf{I}_3=\frac{4}{15}\mu\pi abc(a^2+b^2)$, $\mathbf{I}_4=\frac{4}{15}\mu\pi abc(a^2+c^2)$,

$\mathbf{I}_5=\frac{4}{15}\mu\pi abc(b^2+c^2)$, $\mathbf{I}_6=\frac{4}{15}\mu\pi abc(a^2+b^2+c^2)$,

$P=\frac{1}{15}\mu ab^2c^2$, $P_1=\frac{1}{15}\mu a^2bc^2$, $P_2=\frac{1}{15}\mu a^2b^2c$.

Let $m=n=p=1$. Then for $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{z}{c}\right)^{\frac{2}{3}}=1$,

$\mathbf{I}=\frac{4}{715}\mu\pi a^3bc$, $\mathbf{I}_1=\frac{4}{715}\mu\pi ab^3c$, $\mathbf{I}_2=\frac{4}{715}\mu\pi abc^3$,

$\mathbf{I}_3=\frac{4}{715}\mu\pi abc(a^2+b^2)$, $\mathbf{I}_4=\frac{4}{715}\mu\pi abc(a^2+c^2)$,

$\mathbf{I}_5=\frac{4}{715}\mu\pi abc(b^2+c^2)$, $\mathbf{I}_6=\frac{4}{715}\mu\pi abc(a^2+b^2+c^2)$.

$$P=\frac{64\mu ab^2c^2}{15.13.11.7.5},\quad P_1=\frac{64\mu a^2bc^2}{15.13.11.7.5},\quad P_2=\frac{64\mu a^2b^2c}{15.13.11.7.5}.$$

Thus we could multiply examples without number.

Formulæ (1), (3), (6), (8), (10), (12), (14), (16), (18), will hold for m, n, p fractional as well as integral.

For the radius of gyration we have

$$k_n^2=\frac{I_n}{\mu A},\quad K_n^2=\frac{\mathbf{I}_n}{\mu V}.$$

where A and V are known. (see AMERICAN MATHEMATICAL MONTHLY, page 380, Vol. I., No. 11.)

A SIMPLE DEDUCTION OF THE DIFFERENTIAL OF LOGx.

By J. W. NICHOLSON, A. M., LL. D., Professor of Mathematics in Louisiana State University.

Let $f(x)=\log x$ ....(1), then $f(xy)=f(x)+f(y)$ ....(2).

Differentiate, $f'(xy)(ydx+xdy)=f'(x)dx+f'(y)dy$ ....(3).

Since (3) is true when x and y are independent,

$f'(xy)ydx=f'(x)dx$ .... (4), and $f'(xy)xdy=f'(y)dy$. .... (5).

$$(4)\div(5),\quad \frac{f'(x)}{f'(y)}=\frac{y}{x}=\frac{1/x}{1/y}\ \ldots\ldots\ldots\ldots(6).$$

$\therefore f'(x)=\frac{m}{x}$, $f'(y)=\frac{m}{y}$ ....(7). $\therefore d\log x=\frac{m}{x}dx$ ......(8).

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M. (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society ; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from November Number.]

PROPOSITION XXX. *To any terminated straight AB stands at right angles (Fig. 36.) a certain unbounded straight BX. I say firstly, that the straight AY, erected perpendicularly toward the same parts upon AB, will be one intrinsic limit of all those straights, which drawn from the point A out toward the same parts have (in hypothesis of acute angle) a common perpendicular in two distinct points with the other unbounded straight BX. I say secondly that no acute angle will be the minimum of all, produced under which a straight from the aforesaid point A (in the aforesaid hypothesis) has in two distinct points a common perpendicular with BX.*

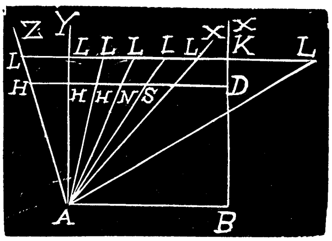

Fig. 36.

Proof of the first part.

For since AY has in common at two distinct points A and B the perpendicular AB with BX; if any straight AZ is drawn toward the same parts under an obtuse angle, it follows there can be toward these parts in two distinct points no common perpendicular to AZ, BX. Otherwise from the resulting quadrilateral containing four angles greater than four right angles, we hit (from Proposition XVI.) upon the already rejected hypothesis of obtuse angle, against the hypothesis of acute angle in this place assumed.

Therefore that perpendicular AY will be from that side an intrinsic limit of all the straights which drawn from the point A toward the same parts have (in the hypothesis of acute angle) at two distinct points a common perpendicular with the other unbounded straight BX. Quod erat primum.

Proof of the second part.

For if it were possible, let a certain acute angle be the least of all, drawn under which AN has with BX in two distinct points the common perpendicular ND. Then in BX a higher point K being assumed, from this erect to BX the perpendicular KL, upon which from the point A let fall (by Euclid I. 12) the perpendicular AL.

But now, if this AL meets ND in any point S, it certainly follows that angle BAL will be less than BAN, which therefore will not be the least of all drawn under which AN has with BX in two distinct points a common perpendicular ND.

But furthermore that the aforesaid perpendicular ND is cut by this perpendicular AL in some intermediate point of it S is thus demonstrated.

And first indeed, that BK cannot be cut by AL in any point M follows absolutely from Euclid I. 17, since otherwise in the same triangle MKL we would have two right angles at the points K and L, apart from the fact that in this case we would have our assertion about that angle BAN, that it is not in such circumstances the least of all.

But again AL cannot be the continuation of AN; because otherwise in the quadrilateral $NDKL$ we would have four right angles, against the hypothesis of acute angle.

But neither can it cut DN produced in any exterior point H; because angle AHN (from Euclid I. 16) would be acute, on account of the external angle AND supposed right; and therefore angle DHL would be obtuse, and so in the quadrilateral $DHLK$ we would have four angles, which taken together would be greater than four right angles, against the aforesaid hypothesis of acute angle.

Therefore it follows that the angle BAN must be cut by this AL, and therefore cannot be declared the least of all, drawn under which AN has with BX in two distinct points a common perpendicular ND.

Quod erat secundo loco demonstrandum. Itaque constat etc.

Corollary. But hence is permitted to observe, that under a lesser angle BAL is obtained (in hypothesis of acute angle) a common perpendicular LK, more remote indeed from the base AB, as follows from the construction, but moreover less than the other nearer common perpendicular ND, which is obtained under a greater angle BAN.

The reason of this latter is because in the quadrilateral $LKDS$ the angle at the point S is acute in the aforesaid hypothesis, since the three remaining angles are supposed right.

Wherefore (from Corollary I. to Proposition III.) the side LK will be less than the opposite side SD, and so much less than the side ND.

[To be Continued.]

SOPHUS LIE'S TRANSFORMATION GROUPS.

A SERIES OF ELEMENTARY, EXPOSITORY ARTICLES.

By EDGAR ODELL LOVETT, Princeton University.

III.

Construction of a One Parameter Group from an Infinitesimal Transformation.

9. Let there be given the one parameter continuous group

$$x_1=\varphi(x, y, a), \quad y_1=\psi(x, y, a); \qquad (1)$$

assume further that it contains the inverse transformation of every transformation in it, *i. e.* that the solutions of the equations (1) with regard to x and y have the form

$$x=\varphi(x_1, y_1, b), \quad y=\psi(x_1, y_1, b),$$

in which b is a canstant depending only on a. In the preceding paragraphs the theorem of Lie that every one parameter group whose transformations are inverse in pairs contains an infinitesimal transformation was arrived at both geometrically and analytically. Either process may be formulated symbolically as follows. If T_a represent the transformation of the group corresponding to the parameter a, its inverse T_a^{-1} is also contained in (1) by hypothesis. Further $T_{a+\delta a}$ will represent the transformation corresponding to the parameter $a+\delta a$, and therefore the transformation of the group (1) that differs from T_a by an infinitesimal. The successive application or the product of $T_{a+\delta a}$ and T_a^{-1}, namely $T_{a+\delta a}T_a^{-1}$ (which belongs to the group by virtue of our first supposition that the product of any two transformations of the group is itself a transformation of the group), differs infinitesimally from $T_aT_a^{-1}$, the identical transformation, and hence is itself an infinitesimal transformation belonging to the group (1).

10. On the other hand there is always a completely determinate continuous group of transformations which contains a given infinitesimal transformation. The truth of this assertion may be made to appear symbolically in the following manner.

Let S be any arbitrary transformation in the xy-plane. Construct the transformations which are equivalent to the repetition of S once, twice, and so on to n-times ; also the inverse of S, S^{-1}, and those equivalent to the repetition of this inverse once, twice, and so on to n-times ; we then have an infinite family of transformations,

$$\ldots, S^{-n}, \ldots, S^{-2}, S^{-1}, S^0, S^1, S^2, \ldots, S^n, \ldots,$$

where S^0 is the identical transformation, while n represents every possible positive whole number. This infinite family is a group, since if p and q are two positive or negative numbers, the product of S^p and S^q is equivalent to S^{p+q}, but the group is a discontinuous one.

In this manner, beginning with an arbitrary transformation S an infinite number of discontinuous groups in x and y may be constructed. Passing now to the limiting case, if, in particular, S is an infinitesimal transformation, then S^n and S^{n+1} differ from each other by an infinitesimal, and we have accordingly a continuous group constructed from, and containing the infiniteslmal transformation, S.

11. Lie has invented an ingenious kinematical illustration of this limiting case, which serves as a concrete introduction to the rigorous demonstration of the theorem.

The infinitesimal transformation is defined by two equations of the form

$$x'=x+\xi(x, y)\delta t+\ldots, \quad y'=y+\eta(x, y)\delta t\ldots, \tag{2}$$

where ξ and η are any two given functions of x and y, the quantity δt an infinitesimal, and the terms omitted convergent power series in δt beginning with δt^2.

The coördinates of the transformed point (x', y') differ from those of the original point (x, y) by the infinitesimal increments

$$\delta x = \xi(x, y)\delta t, \quad \delta y = \eta(x, y)\delta t,$$

when terms of the second order of infinitesimals are neglected. The infinitesimal transformation makes correspond to every point (x, y) an infinitesimal arrow (say) whose length is

$$\sqrt{\delta x^2 + \delta y^2} = \sqrt{\xi^2 + \eta^2}\ \delta t,$$

and direction
$$\frac{\delta y}{\delta x} = \frac{\eta}{\xi};$$

and in general to different points arrows of different lengths and different directions. The infinitesimal transformation thus puts all the points (x, y) of the plane in motion, and if the variable t be taken as the time, these points describe in the element of time δt, the infinitesimal paths $\sqrt{\xi^2 + \eta^2}\ \delta t$, whose projections on the axes are $\xi\delta t$ and $\eta\delta t$. In the first element of time δt the point (x, y) goes over into (x', y') describing the path $\sqrt{\xi(x, y)^2 + \eta(x, y)^2}\ \delta t$, in the next element δt it runs over the infinitesimal path $\sqrt{\xi(x', y')^2 + \eta(x', y')^2}\ \delta t$, and so on. The original point (x, y) assumes, by the continued application of the infinitesimal transformation, a continuous series of positions which may be represented by a curve. This motion of the points of the plane is characterized by the fact that the components of the velocity of every point (x, y) have the values

$$\frac{dx_1}{dt} = \xi(x_1, y_1), \quad \frac{dy_1}{dt} = \eta(x_1, y_1),$$

which depend only on the position and not on the time. Since the change of position is to repeat itself from moment to moment, the motion is a so-called stationary motion and can be compared to the flow of the particles of a compressible fluid. That the phenomena of a stationary motion exhibit the group property is readily seen, for if the stationary motion carries the points (x, y) to the position (x_1, y_1) in the time t_1 and then these new points (x_1, y_1) to the positions (x_2, y_2) in the time t_2, it is clear that the motion carries the original points (x, y) to the positions (x_2, y_2) in the time $t_1 + t_2$; *i. e.* the successive performance of two transformations (t_1) and (t_2) of the family is equivalent to a single transformation $(t_1 + t_2)$ of the family.

12. This kinematical illustration may now be replaced by the following rigorous analytical reasoning.

The two differential equations

$$\frac{dx_1}{dt} = \xi(x_1, y_1), \quad \frac{dy_1}{dt} = \eta(x_1, y_1), \tag{3}$$

determine x_1 and y_1 as functions of t, and the initial values corresponding to $t = 0$

which we take as $x_1=x$, $y_1=y$. In order to determine these functions x_1 and y_1, it is necessary to integrate the simultaneous system

$$\frac{dx_1}{\xi(x_1, y_1)}=\frac{dy_1}{\eta(x_1, y_1)}=dt, \tag{4}$$

with the initial conditions that $x_1=x$ and $y_1=y$ for $t=0$.

This integration is effected as follows: The differential equation in x_1, y_1

$$\frac{dx_1}{\xi(x_1, y_1)}=\frac{dy_1}{\eta(x_1, y_1)}$$

has an integral, $\Omega(x_1, y_1)$, which, since it is free from t, is also an integral of the whole simultaneous system (4). In order to find the second integral of the system which contains t, we eliminate say y_1 between the two equations

$$\Omega(x_1, y_1)=\text{constant}=c, \quad \text{and} \quad \frac{dx_1}{\xi(x_1, y_1)}=dt,$$

and obtain a differential equation,

$$\frac{dx_1}{\theta(x_1, c)}=dt.$$

Since the left hand member of this equation does not contain t it can be integrated by a quadrature* and its integral has the form $f(x_1, c)-t$. But this is not an integral of the system (4) until c has been eliminated by means of the equation $\Omega(x_1, y_1)=c$. Eliminating c, the second integral of the system (4) appears in the form $W(x_1, y_1)-t$.†

Finally, determining the constants of integration by the initial conditions that $x_1=x$, $y_1=y$ for $t=0$, we have as the result of the integration

$$\begin{aligned} \Omega(x_1, y_1)&=\Omega(x, y), \\ W(x_1, y_1)-t&=W(x, y). \end{aligned} \tag{5}$$

Without solving these equations for x_1, y_1 it is easy to see that they define a one parameter group, for the transformation of the family (5) which corresponds to the parameter value t carries the points (x, y) into the points (x_1, y_1), whose coördinates can be found by solving the equations (5) for x_1, y_1. A sec-

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*By the term quadrature is meant an integral of the form $\int F(x)dx$. It is assumed that a quadrature can always be performed.

†The reader will observe that this same integral would have been found had we begun by eliminating x_1 from $\frac{dy_1}{\eta(x_1, y_1)}=dt$ by means of $\Omega(x_1, y_1)=c$. This elimination would have given the differential equation $\frac{dy_1}{\lambda(y_1, c)}=dt$; the integral of the latter, $\mu(y_1, c)-t$, is found by a quadrature; eliminating c by means of $\Omega(x_1, y_1)=c$, we have finally the second integral of the system, $W(x_1, y_1)-t$.

ond transformation of the same family with the parameter value t_1 will change the points (x_1, y_1) into the points (x_2, y_2) whose coördinates are found from the equations,

$$\begin{aligned} \Omega(x_2, y_2) &= \Omega(x_1, y_1), \\ W(x_2, y_2) - t &= W(x_1, y_1). \end{aligned} \tag{6}$$

In order to find the transformation which carries the original points (x_1, y_1) directly into the final positions (x_2, y_2), it is only necessary to eliminate x_1, y_1 from the equations (5) and (6). The elimination gives at once

$$\begin{aligned} \Omega(x_2, y_2) &= \Omega(x, y), \\ W(x_2, y_2) - (t + t_1) &= W(x, y). \end{aligned}$$

But these equations represent the transformation of the family (5) corresponding to the parameter value $t + t_1$; hence the family (5) possesses the group property. The group contains also the inverse transformation of every transformation in it and the identical transformation.

The equations (5) can be solved with regard to x_1, y_1 in the form

$$x_1 = \Phi(x, y, t), \quad y_1 = \Psi(x, y, t). \tag{7}$$

These two functions can be expanded in powers of t by Maclaurin's theorem. In order to effect the expansion we must have the values

$$\left(\frac{dx_1}{dt}\right)_{t=0}, \quad \left(\frac{d^2x_1}{dt^2}\right)_{t=0}, \quad \ldots.$$

From equations (4) we have $\dfrac{dx_1}{dt} = \xi(x_1, y_1)$, with $x_1 = x$, $y_1 = y$, for $t = 0$;

hence, $$\left(\frac{dx_1}{dt}\right)_{t=0} = \xi(x, y).$$

The equations (4) give also

$$\begin{aligned} \frac{d^2x_1}{dt^2} &= \frac{\partial\xi(x_1, y_1)}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial\xi(x_1, y_1)}{\partial y_1}\frac{dy_1}{dt} \\ &= \frac{\partial\xi(x_1, y_1)}{\partial x_1}\xi(x_1, y_1) + \frac{\partial\xi(x_1, y_1)}{\partial y_1}\eta(x_1, y_1); \end{aligned}$$

hence $$\left(\frac{d^2x_1}{dt^2}\right)_{t=0} = \frac{\partial\xi(x, y)}{\partial x}\xi(x, y) + \frac{\partial\xi(x, y)}{\partial y}\eta(x, y).$$

Similarly, $$\left(\frac{dy_1}{dt}\right)_{t=0} = \eta(x, y), \quad \left(\frac{d^2y_1}{dt^2}\right)_{t=0} = \frac{\partial\eta(x, y)}{dx}\xi(x, y) + \frac{\partial\eta(x, y)}{dy}\eta(x, y).$$

Accordingly equations (7) become by Maclaurin's theorem,

$$
\left.\begin{aligned}
x_1&=x+\xi(x,\ y)t+\left(\ \xi\frac{\partial\xi}{\partial x}+\eta\frac{\partial\xi}{\partial y}\right)\frac{t^2}{1.2}+\ldots,\\
y_2&=y+\eta(x,\ y)t+\left(\ \xi\frac{\partial\eta}{\partial x}+\eta\frac{\partial\eta}{\partial y}\right)\frac{t^2}{1.2}+\ldots
\end{aligned}\right\}\quad(8)
$$

The reader will observe that $t=0$ in the equations (8) gives the identical transformation, and $t=\delta t$ gives an infinitesimal transformation which to terms of the second order agrees with the original infinitesimal transformation (2).

All these facts may now be summed up in the following theorem of LIE:

Every infinitesimal transformation

$$x_1=x+\xi(x,\ y)\delta t+\ldots,\quad y_1=y+\eta(x,\ y)\delta t+\ldots,$$

belongs to at least one one parameter group with inverse transformations, when infinitesimals of the second and higher orders are neglected. The finite equations of this group are found by integrating the simultaneous system

$$\frac{dx_1}{\xi(x_1,\ y_1)}=\frac{dy_1}{\eta(x_1,\ y_1)}=dt,$$

with the initial conditions

$$x_1=x,\quad y_1=y,\quad \textit{for}\ t=0,$$

in the form

$$\Omega(x_1,\ y_1)=\Omega(x,\ y),$$
$$W(x_1,\ y_1)-t=W(x,\ y)\ ;$$

or, solved with regard to x, y, and developed in powers of t, in the form

$$x_1=x+\xi\,(x,\ y)\frac{t}{1\,!}+\left(\ \xi\frac{\partial\xi}{\partial x}+\eta\frac{\partial\xi}{\partial y}\right)\frac{t^2}{1\,!}+\ldots,$$

$$y_1=y+\eta(x,\ y)\frac{t}{1\,!}+\left(\ \xi\frac{\partial\eta}{\partial x}+\eta\frac{\partial\eta}{\partial y}\right)\frac{t^2}{2\,!}+\ldots,$$

The one parameter group thus generated accordingly possesses an infinitesimal transformation which in its terms of the first order is identical with the original infinitesimal transformation.

We have now proved that every G_1 contains an infinitesimal transformation and conversely that every infinitesimal transformation generates a G_1. We shall prove in the next article that a G_1 contains but one infinitesimal transformation, with the converse that an infinitesimal transformation belongs to but one G_1. The theorems will be illustrated by concrete examples. These theorems establish the equivalence of the notions one parameter group and infinitesimal transformation; that these notions may be used interchangeably is the fundamental principle of LIE's Theory of the Group of One Parameter.

Princeton University, 14 December, 1897.

[To be Continued.]

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

74. Proposed by NELSON S. RORAY, South Jersey Institute, Bridgeton, N. J.

Solve according to the conditions given :

$$\sqrt{x+1}+\sqrt{x}=\frac{3}{\sqrt{1+x}}.$$

First, square without transposing and then solve ; second, transpose $\sqrt{x+1}$ and then solve. Obtain the same roots as in the first way of solving.

I. Solution by J. M. BOORMAN, Counselor, Inventor, etc., etc., Hewlett, L. I., N. Y.

Solve ("conditions given") $\sqrt{x+1}+\sqrt{x}=\dfrac{3}{\sqrt{1+x}}$. . . . . (A).

The equation is of first degree. $\therefore$ *can have but one root, e. g.*

FIRST. The *conditioned* operation gives, $2(x+1)+2\sqrt{(x+1)}\sqrt{x}$ $=1+\dfrac{9}{1+x}$. Thence $\sqrt{x+1}.\sqrt{x}=-(x+1)+\frac{1}{2}+\dfrac{9}{2(1+x)}$.

Square, etc., and reduce : $\therefore 8\frac{3}{4}(1+x)^2-4\frac{1}{2}(1+x)=20\frac{1}{4}$.

Divide and supply : $\therefore (1+x)^2-\frac{18}{35}(1+x)+\frac{81}{1225}=\frac{2916}{1225}$.

$\therefore x=\frac{4}{5}$; $x_1=-\frac{16}{7}$.

BUT, apply x_1 to the *given equation*. $\therefore \left(\dfrac{3+4}{\sqrt{7}}\right)\sqrt{-1}=\dfrac{3\sqrt{7}}{3\sqrt{-1}}=\dfrac{\sqrt{7}}{\sqrt{-1}}$.

Now $\dfrac{3+4}{\sqrt{7}}=\sqrt{7}$. $\therefore \sqrt{7}\sqrt{-1}=\sqrt{7}\left(\dfrac{1}{\sqrt{-1}}\right)$.

$\therefore -1\sqrt{7}=\sqrt{7}$; or $2\sqrt{7}=0$! ! So $x_1=-\frac{16}{7}$ *is not a root* of equation (A), but of its factor $\sqrt{x+1}+\sqrt{x}=\dfrac{-3}{\sqrt{1+x}}$, that *inevitably* results by the conditioned involution. Hence $x=\frac{4}{5}$ *only*.

SECOND (direct) "way." Transpose and square.

$\therefore x=x+1-6+\dfrac{9}{1+x}$. Thence $x=\frac{4}{5}$, [the "*same root as in the first way.*"]

PROOF. Apply *this* $x=\frac{4}{5}$, in equation (A).

$\therefore \sqrt{\frac{9}{5}}+\sqrt{\frac{4}{5}}=\dfrac{3\sqrt{5}}{\sqrt{9}}$. $\therefore$ as $\dfrac{3+2}{\sqrt{5}}=\dfrac{5}{\sqrt{5}}=\sqrt{5}$, so $\sqrt{5}=\dfrac{3\sqrt{5}}{\sqrt{9}}=\frac{3}{3}\sqrt{5}=1\sqrt{5}$,

i. e. $\sqrt{5}=\sqrt{5}$, *satisfies* equation (A). $\therefore x=\frac{4}{5}$ is the *one* root of (A). Q. E. D.

II. **Solution by J. SCHEFFER, A. M., Hagerstown. Md.**

Squaring the equation as it stands, we get $2x+1+2\sqrt{x(x+1)}=\dfrac{9}{x+1}$.

Clearing of fractions and leaving the radical by itself in the first member, we get $2(x+1)\sqrt{x(x+1)}=8-3x-2x^2$. Squaring, arranging, and cancelling, we get the quadratic $35x^2+52x=64$, the two roots of which are $x=\frac{4}{5}$ and $-\frac{16}{7}$, the the former of which satisfies the equation $\sqrt{x+1}+\sqrt{x}=\dfrac{3}{\sqrt{x+1}}$, and the latter the equation $\sqrt{x+1}-\sqrt{x}=\dfrac{3}{\sqrt{x+1}}$.

Clearing the original equation of its denominator $\sqrt{x+1}$, we have $x+1+\sqrt{x(x+1)}=3$, or $\sqrt{x(x+1)}=2-x$. Squaring, we have $5x=4$. $\therefore x=\frac{4}{5}$.

III. **Solution by F. M. McGAW, A. M., Professor of Mathematics in Bordentown Military Institute, Bordentown. N. J.; CHAS. C. CROSS, Laytonsville, Md.; G. B. M. ZERR. A. M., Ph. D., The Russell College, Lebanon, Va.: and J. P. BURDETTE, Class of '97. Dickinson College. Carlisle, Pa.**

(1), $\sqrt{x+1}+\sqrt{x}=\dfrac{3}{\sqrt{1+x}}$. $2x+1+2\sqrt{x+x^2}=\dfrac{9}{1+x}$.

$\therefore 8-2x^2-3x=2(x+1)\sqrt{x+x^2}$. $\therefore 35x^2+52x=64$. $\therefore x=\frac{4}{5}$, or $-2\frac{2}{7}$.

(2). Regarding $\sqrt{x+1}$ as affected by the $\pm$ sign

$\sqrt{x}=\dfrac{2-x}{\sqrt{1+x}}$ or $\dfrac{4+x}{\sqrt{1+x}}$.

$\therefore x=(4-4x+x^2)/(1+x)$, or $(16+8x+x^2)/(1+x)$.

$\therefore x=\frac{4}{5}$, or $x=-2\frac{2}{7}$.

Also solved by *A. H. BELL*.

75. **Proposed by the late B. F. BURLESON, Oneida Castle, N. Y.**

Mr. B's farm is in shape a quadrilateral, both inscriptible and circumscriptible, and contains an area of $k=10752$ square rods. The square described on the radius of its inscribed circle contains $r^2=2304$ square rods; while the square described on the radius of its circumscribed circle contains an area of $R^2=7345$ square rods. Required the lengths of the sides of his farm.

I. **Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.**

Let a, b, c, d be the sides required. By the conditions of the problem, $a+c=b+d$; $abcd=k^2=115605504$ ....(1).

$\frac{1}{2}r(a+b+c+d)=k$, or $a+b+c+d=2k/r=448$.

$\therefore a+c=b+d=k/r=224$ ....(2).

$$R=\tfrac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(bc+ad)}{abcd}}.$$

$\therefore\ (ab+cd)(ac+bd)(bc+ad)=16R^2k^2=13585958830080 \ldots\ldots\ldots\ldots (3).$

Substituting (2) in (1) and (3), we get

$$(224a-a^2)(224b-b^2)=115605504 \ldots\ldots\ldots\ldots (4).$$

$$\{ab+(224-a)(224-b)\}\{a(224-a)+b(224-b)\}\{b(224-a)+a(224-b)\}$$
$$=13585958830080 \ldots\ldots\ldots\ldots (5).$$

Eliminating b from (4) and (5), we get, after reducing and factoring,

$$(a-168)(a-128)(a-96)(a-56)=0.$$

$\therefore$ The sides are 168, 128, 96, 56 rods, respectively.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.; and COOPER D. SCHMITT, A. M., Professor of Mathematics in the University of Tennessee, Knoxville, Tenn.

Denoting the four consecutive sides of the quadrilateral by a, b, c, d, we have, from well-known geometrical formulæ and principles:

$abcd=k^2 \ldots\ldots (1)$; $r^2=abcd/[a+c]^2=k^2/[a+c]^2 \ldots\ldots (2)$;

$a+c=b+d=k/r \ldots\ldots (3)$; $R^2=\{[ac+bd][ad+bc][ab+cd]\}/16k^2 \ldots\ldots (4)$.

Putting $ac=x$, and $bd=y$, we have in (4),

$[x+y]\{ac[b^2+d^2]+y[a^2+c^2]\}=16k^2R^2$; or

$[x+y]\{x[(k^2/r^2)-2y]+y[(k^2/r^2)-2x]\}=16k^2R^2$; or

$[x+y]\{[x+y][k^2/r^2]-4xy\}=16k^2R^2$, and since $xy=abcd=k^2$,

$[x+y]\{[x+y]-4r^2\}=16R^2r^2$; or, reduced

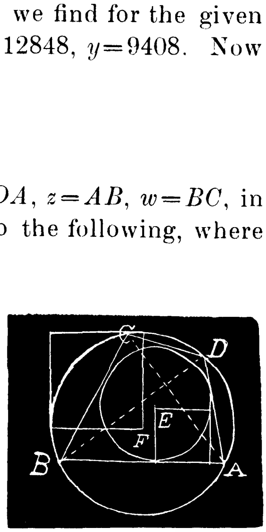
$[x+y]^2-4r^2[x+y]=16R^2r^2$; whence $x+y=2r^2+2r\sqrt{r^2+4R^2}$,

and combining this with $xy=k^2$, we find x and y. Thus, we find for the given numerical values $x+y=21696$, $xy=115605504$, whence $x=12848$, $y=9408$. Now we have $ac=12288$, $a+c=224$, and $bd=9408$, $b+d=224$.

Whence $a=128$, $c=96$, $b=168$, $d=56$.

III. Solution by the PROPOSER.

Let $ABCD$ represent the farm, and let $x=CD$, $y=DA$, $z=AB$, $w=BC$, in order. We have $x+z=y+w \ldots\ldots (1)$. Also the following, where $s=x+y+z+w$, the perimeter of the quadrilateral :



$k^2=\frac{1}{16}\{[s-2x][s-2y][s-2z][s-2w]\} \ldots\ldots (2)$;

$rs=2k \ldots\ldots (3)$; $k^2=xyzw \ldots\ldots (4)$;

$R^2=\{[xy+zw][xz+yw][xw+yz]\}$

$\div\{[s-2x][s-2y][s-2z][s-2w]\} \ldots (5)$.

Substitute in (2), (4), and (5) :

$m=xy+xz+xw+yz+yw+zw$;

$n=xyz+xyw+xzw+yzw$; and $p=xyzw$.

Then we shall have by involving terms and re-factoring,

$16k^2=16p-s^4+4s^2m-8sn$ ....(6) ; $k^2=p$. ....(7) ;

$R^2=16p-s^4+4s^2m-8sn=n^2-4pm+ps^2$ ....(8).

From (3), (6), (7), and (8), we obtain by elimination and resolution,

$s=2k/r=448$; $m=[k^3+r^3n]/r^2k=71872$; $n=2rk+2k\sqrt{[4R^2+r^2]}=4859904$; $p=k^2=115605504$.

We now, by the "Theory of Equations," construct the biquadratic, the four roots of which will be the values of x, y, z, and w.

$x^4-448x^3+71872x^2-4859904x=-115605504$............(9).

The four roots of equation (9), we find to be 56, 96, 128, and 168. Arranging these values in conformity with equation (1), we have, $CD=x=56$ rods, $DA=y=96$ rods, $AB=z=168$ rods, and $BC=w=128$ rods.

IV. **Solution by A. H. BELL, Hillsboro, Illinois.**

Since circumscriptible quadrilaterals have the sums of their opposite sides equal, take $x+y$, $x+z$, $x-y$, $x-z$, for the sides AB, BC, DC, and AD.

$\therefore 2rx=k$, $x=k/2r$ ....(1).

$\overline{BD}^2=[x+y]^2+[x-z]^2-2[x+y][x-z]\cos A$ ....(3).

$\overline{BD}^2=[x+z]^2+[x-y]^2+2[x-y][x+z]\cos A$,

$\{\cos C=\cos[180-A]=-\cos A\}$ ....(4).

$\therefore \cos A=\dfrac{x[y-z]}{x^2-y^2}$. $\sin^2 A=1-\cos^2 A=\dfrac{[x^2-y^2][x^2-z^2]}{[x^2-y^2]^2}$ ....(5).

$\overline{BD}^2=\dfrac{\{[x^2-y^2]+[x^2-z^2]\}[x^2+y^2]}{[x^2-y^2]}$; also $R^2=\dfrac{\overline{BD}^2}{4\sin^2 A}$ ....(6).

$2k=[x+y][x-z]\sin A+[x-y][x+z]\sin A$, or $[x^2-y^2]\sin A=k$........(7).

Substituting the value of $\sin A$, (5) in (7), and

$[x^2-y^2][x^2-z^2]=k^2$ ....(8),

Then (6) becomes, $[x^2-y^2+x^2-z^2][x^4-y^2z^2]=4Rk^2$ ....(9).

Let the product of the opposite sides $v=x^2-y^2$. $\therefore y^2=x^2-v$....(10) ;

and $w=x^2-z^2$. $\therefore z^2=x^2-w$...................................(11).

Then (8), and (1), $vw=k^2=4r^2x^2$.......................................(12) ;

and (9) reduces to $[v+w]^2-4r^2[v+w]=16R^2r^2$..............................(13).

$\therefore\ v+w=2r^2\pm2r\sqrt{4R^2+r^2}$ ....(14).

$(14)^2-4(7)$, etc. $v-w=2r[4R^2+2r^2-4x^2\pm2r\sqrt{4R^2+r^2}]^{\frac{1}{2}}$ ....(15).

$(14)\pm(15)$ after substituting the given values, $v=12288$, and $w=9408$.

(1), (5), and (6) $x=112$, $y=56$, and $z=16$, and the required sides AB, BC, CD, and AD are 168 rods, 128 rods, 56 rods, and 96 rods, respectively.

Also $BD=158.22$ rods.

Also solved by *CHARLES C. CROSS* and *H. C. WILKES*.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

61. Proposed by W. H. CARTER, Professor of Mathematics, Centenary College of Louisiana, Jackson, La.

If $r=a\sin n\theta$ is the polar equation of a curve, show (1) that the curve consists of n or $2n$ loops according as n is an odd or an even integer; (2) that its area is $\frac{1}{4}$ or $\frac{1}{2}$ of the circumscribing circle according as n is an odd or an even integer.

I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

$r=a\sin n\theta$. Let $r=0$, then $\sin n\theta=0$.

$\therefore\ \theta=0,\ 2\pi/n,\ 4\pi/n,\ 6\pi/n,\ 8\pi/n,\ \ldots\ldots$, are the angles at which the the curve cuts the polar axis at the pole.

$dr/d\theta=na\cos n\theta=0$. $\therefore\ \theta=\pi/2n,\ 3\pi/2n,\ 5\pi/2n,\ 7\pi/2n,\ \ldots\ldots$, gives the points where r has its greatest value, namely, $\pm a$.

When n is odd the values of $n\theta$ for the angles $0,\ 2\pi/n,\ 4\pi/n,\ 6\pi/n,\ 8\pi/n,\ \ldots\ldots$, are $0,\ 2\pi,\ 4\pi,\ 6\pi,\ 8\pi,\ \ldots\ldots$

When n is even the values of $n\theta$ for the angles $0,\ 2\pi/n,\ 4\pi/n,\ 6\pi/n,\ 8\pi/n,\ \ldots\ldots$, are $0,\ \pi,\ 2\pi,\ 3\pi,\ 4\pi,\ \ldots\ldots$

$\therefore$ When n is even the polar axis is cut, at the pole, $2n$ times, but only n times when n is odd.

A=area of one loop.

$$A=\tfrac{1}{2}a^2\int_0^{\pi/n}\sin^2 n\theta d\theta,\ =\frac{\pi a^2}{4n}.$$

$$\therefore\ \frac{\pi a^2}{4n}\times n=\frac{\pi a^2}{4},\ \text{for } n \text{ odd};\ \frac{\pi a^2}{4n}\times 2n=\frac{\pi a^2}{2},\ \text{for } n \text{ even}.$$

II. **Solution by E. L. SHERWOOD, A. M., Superintendent of City Schools, West Point, Miss.**

Equation given $\rho=a\sin n\theta$. We may oberve that,

$$\rho=0,\ a,\ 0,\ -a,\ \text{etc., when}$$

$$\sin n\theta=0,\ 1,\ 0,\ -1,\ \text{etc., when}$$

$$n\theta=0,\ \tfrac{1}{2}\pi,\ \pi,\ \tfrac{3}{2}\pi,\ \text{etc., when}$$

$\theta=\{[c/n].\frac{1}{2}\pi\}$, where c is 0, 1, 2, 3, 4, etc., up to $4n$ ($4n$ being determined by $\theta=2\pi$).

The series of values will be as follows:

$$\theta=0.\frac{\pi}{2n},\quad \frac{\pi}{2n},\ 2.\frac{\pi}{2n},\ 3\frac{\pi}{2n}\ \ldots\ldots\ n.\frac{\pi}{2n},\quad [n+1]\frac{\pi}{2n}\ \ldots\ldots\ 2n\frac{\pi}{2n},$$

$$[2n+1]\frac{\pi}{2n}\ \ldots\ldots\ ;$$

If n is *even*, $\rho=0,\ a,\ 0,\ -a\ldots\ldots0\pm a\ldots\ldots0,\ a$.

If n is *odd*, $\rho=0,\ a,\ 0,\ -a\ldots\ldots\pm a,\ 0\ldots\ldots0,\ -a\ldots\ldots$

$$\left\{\begin{array}{cccc} 3n.\frac{\pi}{2n}, & [3n+1]\frac{\pi}{2n} & \ldots\ldots\ldots\ldots & 4n.\frac{\pi}{2n} \\ 0 & \pm a & \ldots\ldots\ldots\ldots & 0 \\ \pm a & 0 & \ldots\ldots\ldots\ldots & 0 \end{array}\right.$$

In each series are $4n$ terms (the first coincides with the last), and $\rho=a$ numerically in $2n$ of them. But when n is odd, the radius vector traces each loop twice for $\pi/2n$ and a is the same point as $\{[2n+1][\pi/2n]\}$ and $-a$.

$\therefore$ There are $2n$ loops when n is even, and n loops when n is odd.

$$\text{Area }=\tfrac{1}{2}\int\rho^2d\theta,\ \text{where } \rho^2=a^2\sin^2 n\theta,$$

$$=\tfrac{1}{2}a^2\int\sin^2 nd\theta,$$

$$=\tfrac{1}{2}a^2\left[\tfrac{1}{2}\theta-\frac{\sin 2n\theta}{4n}\right]_0^{\pi/2n}=\frac{\pi a^2}{8n}\ \text{for } \tfrac{1}{2}\text{ loop},$$

or $\pi a^2/4n$ for an entire loop.

$\therefore$ For $2n$ loops, area$=\pi a^2/2$; and for n loops, area$=\pi a^2/4$.

Also solved by *J. SCHEFFER* and *C. W. M. BLACK*.

62. **Proposed by A. H. HOLMES, Brunswick, Maine.**

A bucket is in the form of a frustum of a cone having its smaller end as a base. It is a inches in diameter at base and b inches in diameter at top, and its perpendicular height is c inches. It contains water the perpendicular height of which is $\frac{1}{2}c$ inches. What is the greatest height, from the plane on which the vessel rests, to which the surface of the water will rise when the bucket is overturned, no allowance being made for the thickness of the material of the bucket?

Partial Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Mass.

Let $AEFL$ be section of frustum. Complete the cone to the apex N. Let NM be axis, HK surface of water, PQ plane on which vessel rests, AB height of surface of water above PQ. Denote $\angle ONA$ by α, ON by l, HK by x, HM by y.

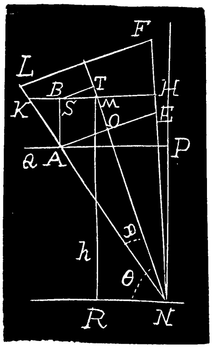


Fig. 1.

Then $EO=OA=\frac{1}{2}a$, $FL=b$. Denote angle which axis makes with PQ, $\angle ONR$ by θ, MR perpendicular to NR by h.

Now as vessel is tipped over, until H reaches E, volume of cone NHK is constant, and $=\frac{1}{3}\pi[\frac{1}{2}a+\frac{1}{6}b)^2[l+\frac{1}{3}c]$. Denote it by C.

Base HK is an ellipse, whose major axis is

$$x=h\cot[\theta-\alpha]-h\cot[\theta+\alpha]\;\ldots\ldots\ldots\ldots(1).$$

Also $HM=y$, $=h\cot\theta+h\cot[\theta+\alpha]$ ....(2).

Let z=semi-minor axis, =ordinate in circular section through S, middle point of HK. Let r=radius of section.

Then $z=\sqrt{r^2-\overline{TS}^2}$, $=\sqrt{r^2-[(x/z)-y]^2\sin^2\theta}$ ....(3),

since $\angle TMS=\angle ONR$, $=\theta$. Also,

$$r=NT\tan\alpha,\ =\left[\frac{h}{\sin\theta}+\left(\frac{x}{2}-y\right)\cos\theta\right]\tan\alpha\;\ldots\ldots(4).$$

Volume $NHK=[\frac{1}{3}\pi]h[x/2]$, $=C$ .... (5).

Let $\cot\theta=\beta$, $\cot\alpha=k$. (1) becomes, $x=\dfrac{2hk[\beta^2+1]}{k^2-\beta^2}$ ....(6);

(2) becomes, $y=\dfrac{h[\beta^2+1]}{k+\beta}$ ....(7), and $\dfrac{x}{2}-y=\dfrac{h\beta\,[\beta^2+1]}{k^2-\beta^2}$ ....(8).

From (4) by (8), $r=\dfrac{hk\sqrt{\beta^2+1}}{k^2-\beta^2}$ ....(9).

Substituting in (3),

$$z=\sqrt{\frac{h^2k^2[\beta^2+1]}{[k^2-\beta^2]^2}-\frac{h^2\beta^2[\beta^2+1]}{[k^2-\beta^2]^2}},\ =h\sqrt{\frac{\beta^2+1}{k^2-\beta^2}}\;\ldots\ldots(10).$$

(5) becomes, $[\frac{1}{3}\pi]h\times\dfrac{hk[\beta^2+1]}{k^2-\beta^2}\times h\sqrt{\dfrac{\beta^2+1}{k^2-\beta^2}}=[\frac{1}{3}\pi]h^3k\left(\dfrac{\beta^2+1}{k^2-\beta^2}\right)^{\frac{3}{2}}=C$;

$$\text{whence } h=\sqrt[3]{\frac{3C}{\pi k}}\sqrt{\frac{k^2-\beta^2}{\beta^2+1}}\;\ldots\ldots\ldots\ldots(11).$$

Now $AB=h-l\sin\theta+\frac{1}{2}a\cos\theta$, $=\sqrt[3]{\frac{3C}{\pi k}}\sqrt{\frac{k^2-\beta^2}{\beta^2+1}}-\frac{l}{\sqrt{\beta^2+1}}+\frac{a\beta}{2\sqrt{\beta^2+1}}$,

$$=\frac{\sqrt[3]{\frac{3C}{\pi k}}\sqrt{k^2-\beta^2}-l+\frac{a\beta}{2}}{\sqrt{\beta^2+1}} \quad \ldots\ldots\ \ldots\ldots\ldots\ldots (12),$$

in which β is the only variable. $dAB/d\beta=$

$$\frac{\sqrt{\beta^2+1}\left(-\sqrt[3]{\frac{3C}{\pi k}}\frac{\beta}{\sqrt{k^2-\beta^2}}+\frac{1}{2}a\right)-\left(\sqrt[3]{\frac{3C}{\pi k}}\sqrt{k^2-\beta^2}-l+\frac{a\beta}{2}\right)\frac{\beta}{\sqrt{\beta^2+1}}}{\beta^2+1}=0$$

Equating to zero, and clearing of fractions,

$$(\beta^2+1)\left(-\sqrt[3]{\frac{3C}{\pi k}}\frac{\beta}{\sqrt{k^2-\beta^2}}+\frac{1}{2}a\right)-\left(\sqrt[3]{\frac{3C}{\pi k}}\sqrt{k^2-\beta^2}-l+\frac{a\beta}{2}\right)\beta=0,$$

$$\text{or } \sqrt[3]{\frac{3C}{\pi k}}\left(\frac{\beta[\beta^2+1]}{\sqrt{k^2-\beta^2}}+\beta\sqrt{k^2-\beta^2}\right)=\frac{1}{2}a+l\beta.$$

Squaring and clearing, $[3C/\pi k]^{\frac{2}{3}}\beta^2[1+k^2]^2=[\frac{1}{2}a+l\beta]^2[k^2-\beta^2]$. Whence,

$l^2\beta^4+al\beta^3+\{[3C/\pi k]^{\frac{2}{3}}[1+k^2]^2+\frac{1}{4}a^2-l^2k^2\}\beta^2-alk^2\beta-\frac{1}{4}a^2k^2=0 \ldots\ldots\ \ldots (13).$

Now $l=\frac{1}{2}a\cot\alpha$, $=\frac{1}{2}ak$; also, $k=\{c/\frac{1}{2}[b-a]\}$, $=\{2c/[b-a]\} \ldots\ldots\ldots (14).$

$\therefore\ l=\dfrac{ac}{b-a}$. Also, $C=\pi\left(\dfrac{2a+b}{6}\right)^2[l+\frac{1}{3}c]$, $=\pi\dfrac{[2a+b]^3c}{108[b-a]}$.

$\therefore\ 3C/\pi k=\{[2a+b]^3/216\} \ldots\ldots\ldots\ldots\ \ldots\ldots\ldots\ldots\ldots\ \ldots\ldots\ldots (16).$

Substituting (14), (15), (16) in (13),

$$\frac{a^2c^2}{[b-a]}\beta^4+\frac{a^2c}{b-a}\beta^3+\left(\frac{[2a+b]^2\{[b-a]^2+4c^2\}}{36[b-a]^2}\right.$$

$$\left.+\frac{1}{4}a^2-\frac{4a^2c^4}{[b-a]^4}\right)\beta^2-\frac{4a^2c^3}{[b-a]^3}\beta-\frac{a^2c^2}{[b-a]^2}=0 \quad \ldots\ \ldots (17).$$

By solving this for β we get the maximum values of AB, provided (Fig. 1) H does not pass E. In Fig. 2, representing this condition $\angle EAB=\theta$,
$\therefore\ AB=a\cos\theta$.

Accordingly (17) will produce critical values of θ, provided $\cos\theta$ is not $>$ AB/a, AB to be determined from (12).

It is evident that for any position of HK which cuts EA, the value of AB will be greater than that determined by the supposition made above. We must seek for maxima in this case by a different method.

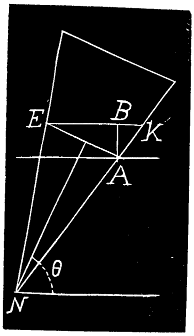


Fig. 2.

DKA (Fig. 3) represents section of volume of water.

$$\text{Volume}=[\pi c/9]\{[\tfrac{1}{4}a^2+[\tfrac{1}{3}a+\tfrac{1}{6}b]^2+\tfrac{1}{2}a[\tfrac{1}{3}a+\tfrac{1}{6}b]\},$$

$$=[\pi c/324][19a^2+7ab+b^2] \quad\ldots\ldots\ldots\ldots(18).$$

Equation (1)—(4) and (6)—(10) apply here as in Fig. 1.

Now volume ADK=cone NDK—cone NDA, $=\tfrac{1}{3}h\times$[area elliptical segment DK]$-\tfrac{1}{3}l\times$[area circular segment AD] ....(19).

Let $AB=s$, $\angle DAC=\theta$, $\angle BKA=\theta-\alpha$.

$$DK=DB+BK,=s\tan\theta+s\cot[\theta-\alpha],=s\left(\frac{1}{\beta}+\frac{k\beta+1}{k-\beta}\right),\ =\left(\frac{sk[1+\beta^2]}{\beta[k-\beta]}\right) \quad\ldots(20).$$

$$\text{Area segment } DK=\frac{\pi zx}{2}-\frac{zx}{2}\left(\cos^{-1}\frac{[DK-\frac{1}{2}x]}{\frac{1}{2}x}-\frac{DK-\frac{1}{2}x}{\frac{1}{2}x}\sqrt{1-\left(\frac{DK-\frac{1}{2}x}{\frac{1}{2}x}\right)^2}\right).$$

Substitute from (6), (10), and (20):

$$\text{Area } DK=\frac{h^2k[\beta^2+1]^{\frac{3}{2}}}{[k^2-\beta^2]^{\frac{3}{2}}}\left[\pi-\cos^{-1}\left(\frac{s[k+\beta]}{h\beta}-1\right)\right.$$

$$\left.+\left(\frac{s[k+\beta]}{h\beta}-1\right)\sqrt{\frac{s[k+\beta]}{h\beta}\left(2-\frac{s[k+\beta]}{h\beta}\right)}\ \right] \quad\ldots.(21).$$

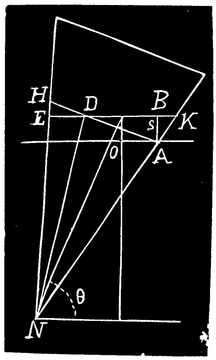


Fig. 3.

Now $s=h-l\sin\theta+\tfrac{1}{2}a\cos\theta$.

$$\therefore\ h=s+l\sin\theta-\tfrac{1}{2}a\cos\theta,\ =s+\frac{2l-a\beta}{2\sqrt{\beta^2+1}} \quad\ldots\ldots(22).$$

$$AD=s\sec\theta,=\frac{s\sqrt{\beta^2+1}}{\beta}.$$

$$\text{Area segment } AD=\tfrac{1}{4}a^2\left[\pi-\cos^{-1}\left(\frac{2s\sqrt{\beta^2+1}}{a\beta}-1\right)\right.$$

$$\left.+\ 2\left(\frac{2s\sqrt{\beta^2+1}}{a\beta}-1\right)\sqrt{\frac{s\sqrt{\beta^2+1}}{a\beta}\left(1+\frac{s\sqrt{\beta^2+1}}{a\beta}\right)}\right] \quad\ldots\ldots.(23).$$

Substitute from (18), (21), (22), and (23) in (19),

$$\frac{k[s\sqrt{\beta^2+1}+l-\tfrac{1}{2}a\beta]^3}{3[k^2-\beta^2]^{\frac{3}{2}}}\left\{\pi-\cos^{-1}\left[\frac{s[k+\beta]}{\beta\{s+[(2l-a\beta)/(2\sqrt{1+\beta^2})]\}}-1\right]\right.$$

$$+\left[\frac{s[+\beta]}{\beta\{s+[(2l-a\beta)/(2\sqrt{\beta^2+1})]\}}-1\right]\times$$

$$\left.\sqrt{\frac{s[k+\beta]}{\beta\{s+[(2l-a\beta)/(2\sqrt{\beta^2+1})]\}}\left[2-\frac{s[k+\beta]}{\beta\{s+[(2l-a\beta)/(2\sqrt{\beta^2+1})]\}}\right]}\right\}$$

$$-\frac{a^3k}{24}\left\{\pi-\cos^{-1}\left[\frac{2s\sqrt{\beta^2+1}}{a\beta}-1\right]\right.$$

$$\left.+2\left[\frac{2s\sqrt{\beta^2+1}}{a\beta}-1\right]\sqrt{\frac{s\sqrt{\beta^2+1}}{a\beta}\left[1-\frac{s\sqrt{\beta^2+1}}{a\beta}\right]}\right\}=\tfrac{1}{324}\pi c[19a^2+7ab+b^2],$$

which equation contains only s, β, and constants. However, the chance of solving it after differentiation seems extremely slight.

PROBLEMS FOR SOLUTION.

MISCELLANEOUS.

56. Proposed by S. HART WRIGHT, A. M., M. D., Ph. D., Penn Yan, N. Y.

In latitude 40° N.$=\lambda$, when the moon's declination is $5^\circ\ 23'$ N.$=\delta$, and the sun's declination $9^\circ\ 52'$ S.$=-\delta'$, how long after sunset will the cusps of the moon's crescent set synchronously, the moon having recently passed its conjunction with the sun ?

57. Proposed by GEORGE LILLEY, Ph. D., LL. D., Professor of Mathematics in the Oregon State University, Eugene, Oregon.

A particle is placed very near the center of a circle, round the circumference of which n equal repulsive forces are symmetrically arranged ; each force varies inversely as the mth power of its distance from the particle. Show that the resultant force is approximately $\dfrac{m_1 n(m-1)}{2r^{m+1}}\times CP$, and tends to the center of the circle, where m_1 is the mass of the particle, CP its distance from the center of the circle, and r the radius of the circle.

EDITORIALS.

The credit of preparing the index for this volume is due Editor Colow.

Dr. Artemas Martin, of the U. S. Coast and Geodetic Survey, has been

promoted to Chief of the Library and Archives Division, at a salary of $1800 per annum, the promotion taking effect July 1, 1897. Dr. Martin has just been elected a member of the "Circolo Matematico di Palermo," Italy.

We regret to announce the death of Prof. B. F. Burleson, which occurred at his home. in Oneida Castle, New York, on December 2. Mr. Burleson was born in Stockbridge, July 7, 1835, but had resided in Oneida Castle for many years, where he was highly esteemed. For a number of years he occupied the position of Principal of the Union School, and in this position he proved a most successful and acceptable teacher. He was extremely fond of mathematics and was very expert in solving difficult problems. For many years he was a frequent contributor to most of the mathematical journals published in this country, and enjoyed a wide acquaintance with well-known mathematical teachers in various parts of the country. Five years ago he was stricken with paralysis and had since suffered from several strokes, which was the final cause of his death. Mr. Burleson was one of those promising but unfortunate men who possessed only the advantages of a common school education. His knowledge of mathematics was obtained by self application and in this way he became a very able analyzer of difficult mathematical problems as his solutions of many difficult problems will show. Had he possessed the advantages of a mathematical course in one of our leading universities, his influence would undoubtedly have been felt in a larger way. All honor is due him for what he made of the opportunities he possessed and the advantages afforded him. There survive him, his widow, a daughter, and one son, George Burleson, of Buffalo, and a sister residing at Oneida Castle.

BOOKS AND PERIODICALS.

Elements of the Differential and Integral Calculus. By William S. Hall, E. M., C. E., M. S., Professor of Technical Mathematics in Lafayette College. 250 pages. Price, $2.25. (1897). New York: D. Van Nostrand.

Great activity has been displayed within the last year in the production of texts on the subject of the Calculus. Among the more recent books on this subject, Professor Hall's treatise is entitled to very favorable consideration. The two branches of the Calculus are treated together to great advantage. The formulas for differentiation are established by the method of limits, but the method of infinitesimals is also explained, and the differential notation used when there is advantage gained by it. The numerical problems illustrating the text and showing applications in engineering practice is an excellent feature of the book. The table of integrals for convenience of reference is more extended than is usual in books of the same scope. Throughout the work there is a great compactness both in the methods and form of treatment, and we find more subjects presented than in most of the elementary texts. The chapter on Differential Equations is one of the best features of the book.

The Calculus for Engineers, with Applications to Technical Problems. By Professor Robert H. Smith.Pages 176. Price, $3.00. 1896. London: Charles Griffin and Company. Philadelphia: J. B. Lippincott Company.

The aim of this treatise is to introduce the student at once to the more important uses of the Integral Calculus, and incidentally to those of the Differential Calculus. The development of the *rattionale* of the subject is based on essentially *concrete* conceptions. Considerable use is made of the graphic method where admissable. The effort has been made to make the treatment less formal than usual, and the meaning and use of results is illustrated by many applications to mechanics, thermodynamics, electrodynamics, problems in engineering design, etc. One of the most distinctive and important features of the book is the very complete and extended Classified Reference Tables of Integrals and Methods of Integration, which occupy 42 pages. The chapter on the integration of Differential Equations will prove an important aid in pointing out methods of dealing with various classes of problems. The book has some practical features that will especially recommend it to engineers and physicists. J. M. C.

The Tutorial Trigonometry. By William Briggs, M. A., F. R. A. S., and G. H. Bryan, Sc. D., F. R. S. London: W. B. Clive. New York: Hinds & Noble. Pages 326. Price, $1.00. 1897.

This latest issue in the series of Tutorial texts is a very satisfactory book. The definitions of the trigonometric functions is wisely introduced early. Most of the articles are written with commendable clearness, and it is only in minor points that we have noticed any defects or inaccuracies in the book. The chapter on the ambiguous case in the solution of triangles is especially clearly and concisely stated. The large number of well-chosen examples attached to each chapter add much to the completeness of the book for class use. The relative importance of subject-matter is indicated by the use of different type, which somewhat mars the appearance of the printed page, but this is a slight objection as compared with the advantage gained in clearness and in effective presentation of the subject to students. While very much after the order of the long list of trigononometries now in use, this book seems to cover about the right ground and bears the marks of a well-constructed text-book. J. M. C.

Regular Points of Linear Differential Equations of the Second Order. By Maxime Bôcher, Ph. D., Assistant Professor of Mathematics in Harvard University. Pages 23. 1896. Cambridge: Harvard University Press.

This excellent little treatise is intended quite as much for students of mathematical physics who may not be able to carry the subject further than is here done as for those intending to make a more extended study of the modern theory of linear differential quotations. J. M. C.

Past and Present Tendencies in Engineering Education. By Mansfield Merriman, Professor of Civil Engineering, Lehigh University, South Bethlehem, Pennsylvania.

This pamphlet of 17 pages, reprinted from Volume IV of the Proceedings of the Society for Promotion of Engineering Education, contains the instructive presidential address of Professor Merriman before that society, at its meeting on August 20, last.

Macfarlane on Discharge of Condenser. This pamphlet contains the interesting discussion of Dr. Macfarlane's paper, which was presented at the meeting of the American Institute of Electrical Engineers in May last, in which Mr. Steinmetz, Dr. Kennelly, and Dr. Perrine took part, and also the communicated reply of Dr. Macfarlane.

Numerical Problems in Plane Geometry. By J. G. Estill, of the Hotchkiss School, Lakeville, Conn. 144 pages. 1897. New York: Longmans, Green & Co.

These problems are meant to be used with other geometries. The book contains a graded set of problems on the five books of geometry, as the division into Books is generally made. The use of the metric system is begun at the very first. The problems, and the entrance papers in the latter part of the book, seem to have been selected with great care and excellent judgment. The discussion of logarithms, and the explanation of their use, and the use of the table, have been clearly made. In as much as some knowledge of the metric system and the ability to solve numerical problems in plane geometry is now required for admission to most colleges, this little treatise should be especially acceptable to preparatory schools. J. M. C.

Euclid: Books I.—IV. By Rupert Deakin, M. A., Headmaster of King Edward's Grammar School, Stourbridge. Price, 70 cents. 1897. London: W. B. Clive. New York: Hinds & Noble.

This edition of Euclid was prepared for the well-known "Tutorial Series." The notes at the end of each book supply excellent comments upon and analysis of the propositions, especially aiding the student to group together propositions in which similar methods of proof are used. Special care has been taken to encourage the working of "riders," and a section is given in which methods of attack are suggested, while exercises on the various methods have been interspersed throughout the text. The book is attractively printed, and should furnish an important aid in teaching elementary Euclid. J. M. C.

School Geometry. By J. Fred Smith, A. M., Principal of Iowa College Academy. 320 pages. 1897. Chicago: Scott, Foresman & Co.

While there is no special novelty or marked improvement in this on other text-books of like purpose and scope, yet it is well written and has several good features. The subject is approached gradually, and as far as may be the abstract through the concrete; it is more elementary than many of the books in common use, and in the earlier part separate figures indicate the successive steps of a construction instead of one figure for all the steps combined. The equation is early introduced and frequently used. Emphasis is placed upon the importance of original work, and a large number of theorems and problems are given as additional exercises. The side references, usual in other books, showing the authority for each step in a demonstration are omitted, but we doubt if this feature will be much in its favor with most teachers. The book is well printed, but not very neatly nor substantially bound. J. M. C.

Infallible Logic. A Visible and Automatic System of Reasoning. By Thomas D. Hawley, of the Chicago Bar. 8vo. 660 pages. Full Leather Binding. Price, $5.00. Chicago: The Dominion Publishing Co.

Standard books are ever welcome when they come to us in forms and bindings representing all the embellishments of the art of bookmaking. Such a book is *Infallible Logic* published by The Dominion Company, Chicago, a copy of which has just come to our desk. The contents are well arranged, the illustrations are fine, the print is clear and neat, and the binding is superb. The Dominion Company is forging ahead as the leading western publishing house making a specialty of fine subscription books. Having salespeople in nearly every nook of the country, the company enjoys a large and growing trade. As this company has a known reputation for liberality towards its agents and fair treatment of them, an agency in this community for the above book, or some other published by this company, would be a source of considerable profit to the one fortunate enough to secure it. Interested readers should write the company for full particulars.

Elementary Arithmetic. By William W. Speer, Assistant Superintendent of Schools, Chicago. 314 pages. 1897. Boston : Ginn & Company.

To the first book of this series we have previously directed special attention. The author emphasizes the importance of early bringing into view the definite relations of quantity. The idea of relative magnitude is made the basis of treatment in this new series of books. Hence simple ratios are made the key to the solution of all problems. The treatise is sufficiently different from others of a similar purpose to give it field for trial, and in the hands of *competent* teachers we predict it will give profitable results.

J. M. C.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.00 per year in advance. Single numbers, 25 cents. The American Monthly Review of Reviews Co., 13 Astor Place, New York.

The January number of the *American Monthly Review of Reviews* is one of the best issues in the history of that magazine. From cover to cover it is thoroughly "live," alert, and forceful. The opening editorial department of "The Progress of the World" gives a clear and exhaustive New Year's summary of political conditions in both hemispheres at the threshold of 1898. The elaborate article on "The Future of Austria-Hungary," by an Austrian, is by all odds the best account yet given in the English language of the warring forces which threaten to undermine the dual monarchy of central Europe; Mr. Charles A. Conant's clean-cut analysis of the present demands for currency reform in the United States is something that no practical man of affairs should fail to read; Dr. W. H. Tolman's summing up of the municipal progress of New York City under Mayor Strong is just what is needed at this time as an encouragement of efforts for civic betterment everywhere; Lord Brassey's remarkable paper on "The Position of the British Navy," with Assistant Secretary Roosevelt's comments, is full of food for thought when read in connection with the compact digest of the United States annual naval report, which follows, and the review of Captain Mahan's new book; two noteworthy letters of Count Tolstoi on the doctrines of Henry George, one addressed to a German disciple of George and the other to a Siberian peasant, are also published in this number. Besides these important and spirited special features, the magazine's regular departments of "Current History in Caricature," "Leading Articles of the Month," "Periodicals Reviewed," and "New Books" cover such timely topics as Hawaiian annexation and the great strike in England.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year in advance. Single number, 10 cents. Irvington-on-the-Hudson.

Among the leading articles in the January number are the following: Stephen Girard and His College; The Real Klondike; Harold Frederick's "Gloria Mandi"; and A Brief History of our late War with Spain.

The Arena. An Illustrated Monthly Magazine. Edited by John Clark Ridpath, LL. D. Price, $2.50 per year in advance. Single number, 25 cents. Boston : The Arena Co.

The Open Court. A Monthly Magazine devoted to the Science of Religion, the Religion of Science, and the Extension of the Religious Parliament Idea. Edited by Dr. Paul Carus ; T. J. McCormack, Assistant Editor ; E. C. Hegeler, and Mary Carus, Associate Editors. Price, $1.00 per year in advance. The Open Court Publishing Co., Chicago, Ill.

The following periodicals have been received : *Journal de Mathématiques Élémentaires*, (1^{er} Decembre 1897) ; *American Journal of Mathematics*, (October, 1897) ; *The Mathematical Gazette*, (October, 1897) ; *L'Intermédiare des Mathématiciens*, (Novembre 1897) ; *Miscellaneous Notes and Queries*, (October, 1897) ; *The Kansas University Quarterly*, (October, 1897) ; *The Monist*, (October, 1897) ; *The Educational Times*, (December, 1897) ; *Science*, (Nos. for the year to September 24, 1897) ; *Bulletin of the American Mathematical Society*, (November, 1897) ; *The Ohio Teacher*, (November, 1897).

SOME ERRATA IN NO. 11.

Page 282, line 2, for "Ay" read Ay^2.
Page 282, line 17, for "$(2x-m^2$" read $(2x-m)^2$.
Page 283, line 5, for "$+b$" read $+b^2$.
Page 286, supply letter D in the figure.
Page 286, line 14, insert *will* before OED.
Page 286, line 5 from bottom, for "$\frac{1}{2}h$" read $\frac{1}{4}h$.
Page 287, line 17, for "27" read 27/2.
Page 288, line 2, for "R^3" read R^2.
Page 288, line 11, for "$\int_{-\sqrt{r^2-x^2}}^{-\sqrt{r^2+x^2}} zdy$" read $\int_o^r dx \int_{-\sqrt{r^2-x^2}}^{+\sqrt{r^2-x^2}} zdy$.
Page 288, line 12, for "$\int_o^s$" read $\int_o^r$.
Page 288, line 18, for "OF" read OB.
Page 288, line 19, for "$(2br/x)$" read $(2br/h)$.
Page 288, line 20, for "O" read 0.
Page 290, last line, for "sume" read sums.